ARITHMETIC

BY

SONNENSCHEIN

AND

NESBITT.

PARTS. II. III.

FRACTIONAL

&

APPROXIMATE

PRICE 3/6

OPINIONS OF THE PRESS ON PART I. (INTEGRAL).

"Mr. Sonnenschein is a pupil, and a thoroughly taught pupil, of Mr. De Morgan's, and it is scarcely necessary to say more in order to convince all who know Mr. De Morgan's works that there is nothing like half-digested work in this arithmetic. This first part of Mr. Sonnenschein's book is admirable of its kind, and better fitted for ordinary school use than Mr. De Morgan's *Arithmetic*, which is more suitable to students and teachers. Brevity and lucidity in the exposition of principle are its main characteristics as a scientific book; and great care in the explanation of simple practical rules for shortening or verifying calculations is its main characteristic in reference to the art of computation. It gives a clear proof of all the rules,—insisting on the exact meaning of the various operations and their interpretation,—and contains a remarkably good chapter on the general properties of numbers, so far as they can be explained to beginners who have only mastered the arithmetic of integers. It is hardly possible to speak too well of this little book, which we have examined very carefully."—*Spectator.*

"Forty years have elapsed since the appearance of Prof. De Morgan's 'Elements of Arithmetic,' at a time when perhaps few teachers, as they submitted the rules of the science to their pupils, cared to establish them upon reason and demonstration. The effect of this work was that a rational arithmetic began to be taught generally, and the mere committing of rules to memory took its due subordinate position in the course of instruction. Such a method of treatment will go far to develop and exercise the reasoning powers, and in the case of many pupils, there is hardly any other subject which can so well be made a groundwork for the exercise of the reasoning faculty. The book before us is avowedly drawn up in agreement with the principles of Mr. De Morgan's work, and the aim of the authors is to lead the student 'to the discovery of the several rules by some path such as an original discoverer might have travelled.' In this first part, which treats of Integral Arithmetic, we consider that they have carried out their principles successfully, and hope they will succeed as well with the remaining two parts, which are to embrace respectively Vulgar Fractions and Approximate Calculations. The rules enunciated are few and tersely given; there is a great store of illustration; elementary difficulties are well stated and honestly grappled with, and cleared up in a way that brings the subject to the level of the capacities of junior students; at the same time advanced as well as young teachers may gather much that is useful from the book. A reader who has carefully gone through the work, can hardly fail to master the early details of the science; if he fail, it will not be the fault of the authors. The subjects treated of are numeration, modes of computation, the so-called first four rules, contracted operations, scales of notation, and properties of numbers. Under this last division we have much valuable matter grouped under the several heads of

Divisibility of Numbers, Casting out Nines, Resolution into Prime Factors, Greatest Common Measure, and Least Common Multiple. Throughout and at the end of the work occur numerous examples, very varied, all of which are carefully arranged, and many fully worked out in two or more ways. With this short analysis of the contents, we heartily commend the work to teachers generally, assuming, of course, that they will regulate their use of it in proportion to the requirements of age and ability of their pupils. The work is neatly got up, and we have detected hardly any errata."—*Nature.*

"Decidedly the clearest, most useful, and easiest method we have seen of teaching the principles and practices of arithmetic. It combines the excellences of Colenso, De Morgan, and Barnard Smith, with excellences peculiar to itself."—*The Rock.*

"It is a very original and well-reasoned system of educating the mind by means of numbers. The authors, working upon the principle 'that the student must be led to the *discovery* of the several rules by some path such as an original discoverer might have travelled,' have really begun at the beginning and logically deduced one step from another, making all so clear as they proceed, that the merest beginner should understand not merely the 'how' but the 'why.' It is perfectly true that more time and space than some may judge necessary are occupied in presenting what is merely one and the same fact under different aspects, and that what appears to be a complete system of arithmetic may be, and often is, packed into less space than the volume before us, which is but the first of three parts. Those, however, who have any experience in teaching, or perhaps remember their own difficulties in working by rule of thumb, will entirely agree with the authors that progress is not mere advance from rule to rule. Any process once properly realized can never be quite forgotten, and to impart to students what our authors term 'a thorough and all but visual realization of each process,' should be the aim of every teacher. They can scarcely have a more efficient book to work with than that of Messrs. Sonnenschein and Nesbitt."—*Standard.*

"This is a work on Arithmetic of a peculiar, and in some respects an original character. Following in the steps of Professor De Morgan, the chief aim of the authors in explaining the *rationale* of the various arithmetical processes is not to give logical demonstrations of the several rules which a student is required to learn, but to carry him along some such path of reasoning as must have been travelled by an original discoverer; the present concise and conventional processes being unravelled, so to speak, and traced up to their first principles. Rules, as they stand in books, express the results of thought, and it is a mental exercise of some value to follow out the course by which these results were in all probability originally arrived at. There is much to recommend in this view of teaching arithmetic; for, as the authors remark in the preface, no subject is so well fitted as this for the early training of the reasoning powers, 'principally because the student is enabled, without apparatus of any kind, steadily to test all his *à priori* conclusions by the light of experience.' In history, physics, and even in language, the student must have premisses supplied him; but his mathematical studies can all be 'evolved from his inner consciousness.'"—*Educational Times.*

THE SCIENCE AND ART

OF

ARITHMETIC;

For the Use of Schools.

PART II. VULGAR FRACTIONS.
PART III. APPROXIMATE CALCULATIONS.

BY

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AND

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"The mills of God grind slowly, but they grind exceeding small."

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PREFACE.

The Second and Third Parts of our Arithmetic, bound together in the present Volume, treat of Fractional quantities in a two-fold aspect. In the Second Part the results are obtained by means of Vulgar Fractions, and are, often at the cost of much labour, "needlessly accurate." In the Third Part, calculations are made with Decimal Fractions, which, in the large majority of cases, save a great deal of labour by systematically disregarding minute quantities. We have thus the power of obtaining accuracy to within any assigned degree, which is all that is needed in the affairs of life. To use, however, such methods without lapsing into slovenliness of thought, it has been necessary to introduce at once the notion of LIMITS. It is evident that in so elementary a treatise as this, hardly more could be done than indicate the existence in the human mind of such notions; but not only is no harm done, but great good is gained, by opening vistas of thought to the young student, and by carefully impressing upon him that in science there is no such thing as finality.

This consideration also must be our justification both for the introduction and for the superficial treatment of Converging Fractions (Part III. Ch. I.): for their introduction, because they furnish the easiest conception of APPROXIMATION; for their superficial treatment, because a knowledge of Algebra is required for their thorough comprehension, and they have on the whole very little bearing on the ultimate objects of the book. This Chapter may therefore be omitted at discretion.

Teachers are often in doubt as to the proper sequence of subjects to be taught. It has been suggested that Decimals ought to be taught before Vulgar Fractions, being so much more easy to manipulate. This theory is very tempting, but fails in the cardinal point that the notion of Decimals is a much later one than that of Vulgar Fractions. The binary divisions are in fact the earliest with which the human mind is acquainted. Moreover, the operations of Decimals are mere applications of rules previously established in Vulgar Fractions. Who, for example, could realize that $\cdot 2 \times \cdot 3 = \cdot 06$, if he had not previously understood multiplication by Vulgar Fractions? The principle of appealing to the experience of the student in order to establish the rule, has induced us to interweave mental arithmetic exercises in the earlier Chapters on Vulgar Fractions. Proportion is treated as a later notion of Fractions, which in fact it is, so long as we confine ourselves to numbers.

The methods we give for the ready Decimalization of English Money, Weights, and Measures, are, with a slight exception, entirely new. They yield, we believe, all the advantages to be derived from the introduction of Decimal Coins, Weights, and Measures, without incurring the fearful inconveniences of a change, or forfeiting the very notable benefits conferred by the sub-division of the English sovereign into $20 \times 12 \times 4$ farthings. As long as there are twelve months in the year, and as long as men prefer to deal in dozens rather than in tens, so long will 12 be a most convenient factor in our coin. Whether our weights and measures ought to be superseded in favour of a decimal system, may be fairly considered an open question; but till such change is effected, the methods of Part III. Ch. VIII., will very nearly answer the same purpose.

A. SONNENSCHEIN.
H. A. NESBITT.

SEPT. 3RD, 1870.

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ERRATA.

P. 12, Ex. F. (18), read $\frac{1}{3} \div 4$.

P. 45, last line but 1, for $1\frac{1}{2}\frac{3}{5}$ read $1\frac{2}{2}\frac{3}{5}$.

P. 113, line 3, read 1 GRAMME = ·0022046212 lbs. av. = 15·4323487 grains.
Hence 1 kilogramme = 2·2046212 lbs. av., altering the other weights in the same proportion.

P. 114, line 15, read $\dot{4}\dot{5}\dot{4}\dot{5} \times 17$.

P. 156, § 18, for Serré, read Serret.

ARITHMETIC.

PART II. VULGAR FRACTIONS.

CHAPTER I.

UNIFORM DENOMINATORS.

§ 1. THE word Fraction is derived from the Latin *fractus*, broken, and Fractional differs from Integral Arithmetic in that it deals with parts of things instead of with whole things, but, as we shall see, the same notions of More and Less still apply.

§ 2. If a whole is divided into two equal parts, each of them is called *one-half;* and therefore there are two halves in a whole.

If a whole is divided into three equal parts, each part is called *one-third*, and therefore there are three thirds in a whole.

If a whole is divided into four equal parts, each part is called *one-fourth*, or *one-quarter*, and therefore there are four fourths or quarters in a whole.

Similarly a whole may be divided into seven sevenths, twenty-one twenty-firsts, thirty-two thirty-seconds, &c.

EXERCISE I.

Find one-half, one-third, one-quarter, one-fifth, one-sixth, one-seventh, one-eighth, one-ninth, one-tenth, one-eleventh and one-twelfth of £57. 15*s.*

§ 3. One-half, one-third, &c., are written, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, &c.

Two-thirds of a whole are obtained by dividing the whole into three equal parts, and taking two of them.

Three-quarters of a whole are obtained by dividing the whole into four equal parts, and taking three of them.

Five-sevenths of a whole are obtained by dividing the whole into seven equal parts, and taking five of them, and so on.

These parts are written, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{7}$, &c.

The lower figure of every fraction, then, shews us into how many equal parts the whole has been divided, and might therefore be called the *Divisor;* but by shewing us the number of parts into which the whole has been divided, it also shews us the *Name* of each part, and is therefore called the NAMER, or, more commonly, the DENOMINATOR.

The upper figure of every fraction shews us *how many* of these parts we take, and is therefore called the COUNTER, or, more commonly, the NUMERATOR.

N.B. The German words *Zähler* and *Nenner* give the same ideas in more familiar language. In German, the ordinal *fünfte* is clearly distinguished from the fractional *Fünftel;* but unfortunately in English the word *fifth* has both these meanings, which sometimes requires to be pointed out.

The words Numerator and Denominator are derived from the Latin *numerare,* to count, and *nominare,* to name.

Example. Find $\frac{2}{3}$ of £157. 4*s.* $5\frac{1}{4}$*d.*

$$\frac{1}{3} \text{ of } £157.\ 4s.\ 5\tfrac{1}{4}d. = 3\underline{)157 \quad 4 \quad 5\tfrac{1}{4}}$$
$$52 \quad 8 \quad 1\tfrac{3}{4}$$
$$\underline{\qquad\qquad 2}$$
$$\frac{2}{3} \text{ of } \quad ,, \quad = £104 \quad 16 \quad 3\tfrac{1}{2}$$

EXERCISE II.

		£.	s.	d.			£.	s.	d.
Find (1)	$\frac{2}{3}$ of	197	15	$4\frac{1}{2}$	Find (7)	$\frac{4}{9}$ of	8043	17	3
(2)	$\frac{3}{4}$ of	1037	17	6	(8)	$\frac{7}{10}$ of	27419	12	$8\frac{1}{2}$
(3)	$\frac{2}{5}$ of	23	7	$9\frac{3}{4}$	(9)	$\frac{5}{11}$ of	489	18	$11\frac{1}{4}$
(4)	$\frac{5}{6}$ of	1001	15	0	(10)	$\frac{11}{12}$ of	3584	16	3
(5)	$\frac{3}{7}$ of	893	1	$9\frac{3}{4}$	(11)	$\frac{7}{15}$ of	23	7	$9\frac{3}{4}$
(6)	$\frac{5}{8}$ of	347	13	6	(12)	$\frac{11}{18}$ of	65	17	9

EXERCISE A.*

(1) A journey is divided into two equal parts; what will each part be called? *Ans.* One half of the journey.

(2) A goose weighs 7 lbs. and half its own weight; what is the weight of the goose? *Ans.* 14 lbs.

* All Exercises numbered alphabetically are to be worked mentally.

(3) A post is buried one half in the ground, and there are five feet above ground; what is the length of the post? *Ans.* 10 feet.

(4) A post is driven through the water into the ground below; $\frac{1}{3}$ is in the ground, $\frac{1}{3}$ under water, and 10 feet above the water. What is the length of the whole post? *Ans.* 30 feet.

(5) Divide the half of 6*d.* among 3 children; what will each child get? *Ans.* 1*d.*

(6) How much is $\frac{2}{3}$ of 1*s.*? *Ans.* 8*d.*

(7) Out of 6*d.* I spend $\frac{1}{3}$ of 1*s.*; how much is left me? *Ans.* 2*d.*

(8) Find the difference between $\frac{2}{3}$ of 1*s.* and $\frac{1}{2}$ of 1*s.* *Ans.* 2*d.*

(9) From $\frac{3}{4}$ of £1 take $\frac{1}{2}$ of 1*s.* *Ans.* 14*s.* 6*d.*

(10) From $\frac{3}{4}$ of £1 take $\frac{2}{3}$ of a guinea. *Ans.* 1*s.*

(11) I sold $\frac{5}{6}$ of a dozen of wine; how many bottles were left? *Ans.* 2 bottles.

(12) To $\frac{3}{5}$ of £1 add $\frac{5}{6}$ of 1*s.* *Ans.* 12*s.* 10*d.*

(13) How much is $\frac{2}{5}$ of an hour? *Ans.* 24 minutes.

(14) How would you get $\frac{7}{11}$ of a thing?
Ans. Divide the whole into 11 equal parts and take 7 of them.

(15) If I cut off $\frac{4}{7}$ of a thing, what part of the whole will be left? *Ans.* $\frac{3}{7}$ of the whole.

(16) Distribute $\frac{4}{7}$ of a guinea among 6 persons. *Ans.* 2*s.* each.

(17) How much is $\frac{5}{12}$ of a yard? *Ans.* 1 ft. 3 in.

Exercise III.

(1) Find the length of $\frac{4}{5}$ of a mile.

(2) Divide $\frac{7}{11}$ of a mile into 8 equal parts.

(3) Find $\frac{4}{7}$ of a ton.

(4) Find $\frac{3}{8}$ of 1 lb. troy.

(5) Find the value of £$\frac{4}{5}$ + £$\frac{2}{3}$ + $\frac{5}{6}$ of 1*s.*

(6) Find the value of $\frac{2}{3}$ of a guinea + £$\frac{1}{3}$ — $\frac{3}{4}$ of 1*d.*

(7) Distribute $\frac{5}{7}$ of 1 cwt. of coals among four persons.

(8) Divide the distance of $\frac{5}{11}$ of 7 miles into 8 equal portions.

(9) If I spend $\frac{2}{3}$ of a guinea per day, how much is that in 10 days?

(10) Find the value of the whole amount, if $\frac{4}{5}$ of it is £429. 6*s.* 8*d.*

(11) Find the value of £$\frac{1}{2}$, £$\frac{1}{3}$, £$\frac{2}{3}$, £$\frac{1}{4}$, £$\frac{3}{4}$, £$\frac{1}{5}$, £$\frac{2}{5}$, £$\frac{3}{5}$, £$\frac{4}{5}$, £$\frac{1}{6}$, £$\frac{5}{6}$, £$\frac{1}{8}$, £$\frac{3}{8}$, £$\frac{5}{8}$, £$\frac{7}{8}$, £$\frac{1}{10}$, £$\frac{1}{12}$, £$\frac{1}{15}$, £$\frac{1}{16}$, £$\frac{1}{20}$.

(12) Find the value of $\frac{1}{2}$ of 1*s.*, $\frac{1}{3}$ of 1*s.*, $\frac{2}{3}$ of 1*s.*, $\frac{1}{4}$ of 1*s.*, $\frac{3}{4}$ of 1*s.*, $\frac{1}{6}$ of 1*s.*, $\frac{5}{6}$ of 1*s.*, $\frac{1}{8}$ of 1*s.*, $\frac{3}{8}$ of 1*s.*, $\frac{5}{8}$ of 1*s.*, $\frac{7}{8}$ of 1*s.*, $\frac{1}{12}$ of 1*s.*

§ 4. We know that $\frac{4}{5}$ means a whole divided into 5 equal parts, of which we take 4; similarly $\frac{5}{5}$ means a whole divided into 5 parts, of which we take all 5, that is, $\frac{5}{5}$ is the whole; but what is the meaning of $\frac{6}{5}$? We must evidently extend our definition of a fraction. $\frac{6}{5}$ must mean that *more wholes than one* are divided into 5 equal parts *each*, and that we take 6 such parts.

$\frac{2}{2}=\frac{3}{3}=\frac{4}{4}=\frac{5}{5}=\frac{6}{6}$, &c. $=1$;

$\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$ are each less than unity;

$\frac{5}{5}=$ unity, and $\frac{6}{5}$, $\frac{7}{5}$, $\frac{8}{5}$, &c., are each greater than unity; hence,

If the numerator is less than the denominator, the value of the fraction is less than unity.

If the numerator equals the denominator, the value of the fraction equals unity.

If the numerator is more than the denominator, the value of the fraction is more than unity.

Fractions which are less than unity are called *Proper Fractions.*

Fractions which are equal to or more than unity are called *Improper Fractions.*

Quantities consisting of an integer *and* a fraction are called *Mixed Numbers*; e.g. $3\frac{1}{2}$, $5\frac{2}{3}$, $114\frac{17}{19}$. (These are read three and a half; five and two-thirds, &c.)

EXERCISE B.

(1) How many halves in $1\frac{1}{2}$?	*Ans.* 3 halves.
(2) How many halves in $5\frac{1}{2}$?	" 11 halves.
(3) How many thirds in $3\frac{2}{3}$?	" 11 thirds.
(4) How many thirds in 5 wholes?	" 15 thirds.
(5) How many quarters in $2\frac{3}{4}$?	" 11 quarters.
(6) How many wholes in 10 halves?	" 5 wholes.
(7) How many halves in 10 wholes?	" 20 halves.
(8) How many wholes in 17 halves?	" $8\frac{1}{2}$.
(9) How many wholes in 17 thirds?	" $5\frac{2}{3}$.
(10) How many wholes in 20 fifths?	" 4 wholes.

(11) Reduce to whole or mixed numbers :

$\frac{13}{6}$.	*Ans.* $2\frac{1}{6}$.	$\frac{25}{8}$.	*Ans.* $3\frac{1}{8}$.	$\frac{35}{9}$.	*Ans.* $3\frac{8}{9}$.
$\frac{15}{4}$.	„ $3\frac{3}{4}$.	$\frac{29}{7}$.	„ $4\frac{1}{7}$.	$\frac{35}{10}$.	„ $3\frac{5}{10}$.
$\frac{11}{9}$.	„ $1\frac{2}{9}$.	$\frac{30}{10}$.	„ 3.	$\frac{35}{12}$.	„ $2\frac{11}{12}$.
$\frac{17}{5}$.	„ $3\frac{2}{5}$.	$\frac{30}{11}$.	„ $2\frac{8}{11}$.	$\frac{35}{3}$.	„ $11\frac{2}{3}$.

(12) Reduce to improper fractions :

$9\frac{3}{8}$.	*Ans.* $\frac{75}{8}$.	$8\frac{4}{7}$.	*Ans.* $\frac{60}{7}$.	$6\frac{7}{8}$.	*Ans.* $\frac{55}{8}$.
$10\frac{4}{5}$.	„ $\frac{54}{5}$.	$5\frac{7}{12}$.	„ $\frac{67}{12}$.	$11\frac{2}{3}$.	„ $\frac{35}{3}$.
$6\frac{4}{11}$.	„ $\frac{70}{11}$.	$10\frac{7}{18}$.	„ $\frac{187}{18}$.	$4\frac{4}{9}$.	„ $\frac{40}{9}$.
$5\frac{4}{5}$.	„ $\frac{29}{5}$.	$4\frac{5}{6}$.	„ $\frac{29}{6}$.	$3\frac{2}{3}$.	„ $\frac{11}{3}$.

(13) I distributed seven apples among some children, giving to each $\frac{1}{3}$ of an apple. How many children were there?

Ans. 21 children.

(14) Which is greater, and by how much, $\frac{7}{5}$ or 2 wholes?

Ans. 2 wholes are greater by $\frac{3}{5}$.

(15) If I cut up $8\frac{2}{5}$ yards of tape into strips of $\frac{1}{5}$ yard each, how many strips shall I get? *Ans.* 42 strips.

§ 5. Reduce the mixed number $8\frac{5}{13}$ to an improper fraction. One whole $=\frac{13}{13}$; eight wholes $=\frac{104}{13}$, and the $\frac{5}{13}$ make $\frac{109}{13}$.

Exercise IV.

Reduce to improper fractions :

(1) $2\frac{1}{2}$.	(6) $84\frac{17}{20}$.	(11) $41\frac{583}{1000}$.	(16) $5000\frac{83}{5000}$.
(2) $3\frac{1}{3}$.	(7) $864\frac{13}{97}$.	(12) $41\frac{83}{1000}$.	(17) $100\frac{28}{33}$.
(3) $7\frac{2}{3}$.	(8) $46\frac{7}{10}$.	(13) $41\frac{3}{1000}$.	(18) $10000\frac{14}{15}$.
(4) $8\frac{3}{4}$.	(9) $46\frac{17}{100}$.	(14) $41\frac{3}{10000}$.	(19) $3001\frac{83}{700}$.
(5) $12\frac{13}{15}$.	(10) $46\frac{7}{100}$.	(15) $400\frac{127}{400}$.	(20) $73\frac{72}{99}$.

Exercise C.

(1)	How many wholes in 3 halves?	*Ans.* $1\frac{1}{2}$.
(2)	„ „ 5 halves?	„ $2\frac{1}{2}$.
(3)	„ „ 5 thirds?	„ $1\frac{2}{3}$.
(4)	„ „ 5 quarters?	„ $1\frac{1}{4}$.
(5)	„ „ 7 quarters?	„ $1\frac{3}{4}$.
(6)	„ „ 11 quarters?	„ $2\frac{3}{4}$.
(7)	„ „ 11 fifths?	„ $2\frac{1}{5}$.

(8) How many wholes in 15 fifths? *Ans.* 3.

(9) Find the difference between 5 wholes and 18 fifths. „ $1\frac{2}{5}$.

(10) If I spend $\frac{1}{7}$ of a shilling in a day, what shall I spend in 14 days? *Ans.* 2*s.*

(11) If I have 32-fifths of a cake, and wish to give a whole cake to each child, to how many children can I give it?

Ans. 6 children and $\frac{2}{5}$ of a cake over.

(12) Reduce to whole or mixed numbers:

$\frac{15}{2}$.	*Ans.* $7\frac{1}{2}$.	$\frac{65}{12}$.	*Ans.* $5\frac{5}{12}$.	$\frac{100}{10}$.	*Ans.* 10.
$\frac{19}{3}$.	„ $6\frac{1}{3}$.	$\frac{11}{10}$.	„ $1\frac{1}{10}$.	$\frac{109}{10}$.	„ $10\frac{9}{10}$.
$\frac{27}{4}$.	„ $6\frac{3}{4}$.	$\frac{27}{10}$.	„ $2\frac{7}{10}$.	$\frac{117}{10}$.	„ $11\frac{7}{10}$.
$\frac{37}{5}$.	„ $7\frac{2}{5}$.	$\frac{39}{10}$.	„ $3\frac{9}{10}$.	$\frac{128}{10}$.	„ $12\frac{8}{10}$.
$\frac{37}{6}$.	„ $6\frac{1}{6}$.	$\frac{48}{10}$.	„ $4\frac{8}{10}$.	$\frac{139}{10}$.	„ $13\frac{9}{10}$.
$\frac{37}{7}$.	„ $5\frac{2}{7}$.	$\frac{56}{10}$.	„ $5\frac{6}{10}$.	$\frac{139}{100}$.	„ $1\frac{39}{100}$.
$\frac{37}{8}$.	„ $4\frac{5}{8}$.	$\frac{65}{10}$.	„ $6\frac{5}{10}$.	$\frac{428}{100}$.	„ $4\frac{28}{100}$.
$\frac{40}{9}$.	„ $4\frac{4}{9}$.	$\frac{74}{10}$.	„ $7\frac{4}{10}$.	$\frac{4089}{100}$.	„ $40\frac{89}{100}$.
$\frac{45}{9}$.	„ 5.	$\frac{83}{10}$.	„ $8\frac{3}{10}$.	$\frac{4089}{1000}$.	„ $4\frac{89}{1000}$.
$\frac{50}{9}$.	„ $5\frac{5}{9}$.	$\frac{92}{10}$.	„ $9\frac{2}{10}$.	$\frac{4089}{4089}$.	„ 1.

§ 6. Reduce $\frac{299}{23}$ and $\frac{529}{29}$ to whole or mixed numbers. Since there are 23 twenty-thirds or 29 twenty-ninths in every whole, *as many times* as we can get 23 twenty-thirds out of 299 twenty-thirds, or 29 twenty-ninths out of 529 twenty-ninths, so many wholes will there be. Divide therefore the numerator by the denominator; the quotient will be wholes, and the remainder, if any, will be fractional parts with the same denominator as the proposed fraction.

23) 299 (13	29) 529 (18
69	239
..	7
$\frac{299}{23}=13$	$\frac{529}{29}=18\frac{7}{29}$

EXERCISE V.

Reduce to whole or mixed numbers:

(1) $\frac{19}{2}$.	(6) $\frac{529}{23}$.	(11) $\frac{158}{10}$.	(16) $\frac{648597}{10}$.
(2) $\frac{43}{3}$.	(7) $\frac{419}{21}$.	(12) $\frac{519}{10}$.	(17) $\frac{648597}{100}$.
(3) $\frac{59}{4}$.	(8) $\frac{584}{17}$.	(13) $\frac{8431}{10}$.	(18) $\frac{648597}{1000}$.
(4) $\frac{50}{5}$.	(9) $\frac{3519}{467}$.	(14) $\frac{12594}{10}$.	(19) $\frac{648597}{10000}$.
(5) $\frac{137}{5}$.	(10) $\frac{37}{10}$.	(15) $\frac{300000}{10}$.	(20) $\frac{648597}{100000}$.

§ 7. Addition and Subtraction of Fractions. Let us recal the meaning of the symbol $\frac{3}{4}$. We have said above that it means 3 pieces of equal size, each of which is called or *named* a quarter. Similarly, $\frac{4}{7}$ means 4 equal pieces, of which each is called a seventh. Note that the symbol 4 performs very different functions above and below the line. Above the line it represents quantity or *number;* below the line, *name.*

"In Addition and Subtraction we must always have the *same kind* of units, viz., so many things of one kind added to or taken from so many things of the *same* kind." (Part I. p. 59.) It follows that we can only add or subtract fractions of the same denomination.

Exercise D.

(1) $\frac{1}{2}+\frac{3}{2}$.	*Ans.* 2.	(11) $2\frac{2}{5}-\frac{4}{5}$.	*Ans.* $1\frac{3}{5}$.
(2) $\frac{2}{3}+\frac{2}{3}+\frac{1}{3}$.	" $1\frac{2}{3}$.	(12) $1-(\frac{2}{5}+\frac{1}{5})$	" $\frac{2}{5}$.
(3) $\frac{1}{4}+\frac{1}{4}+\frac{3}{4}$.	" $1\frac{1}{4}$.	(13) $2\frac{3}{5}+3\frac{4}{5}$.	" $6\frac{2}{5}$.
(4) $\frac{5}{2}+\frac{4}{2}$.	" $4\frac{1}{2}$.	(14) $\frac{4}{5}+\frac{7}{5}$.	" $2\frac{1}{5}$.
(5) $\frac{7}{2}-\frac{4}{2}$.	" $1\frac{1}{2}$.	(15) $1\frac{2}{5}-\frac{4}{5}$.	" $\frac{3}{5}$.
(6) $1\frac{1}{2}+2\frac{1}{2}$.	" 4.	(16) $1\frac{7}{8}+\frac{3}{8}$.	" $2\frac{2}{8}$.
(7) $1\frac{1}{3}+2\frac{2}{3}$.	" 4.	(17) $1\frac{7}{8}-\frac{5}{8}$.	" $1\frac{2}{8}$.
(8) $3\frac{1}{4}+\frac{3}{4}$.	" 4.	(18) $2\frac{9}{10}-1\frac{7}{10}$.	" $1\frac{2}{10}$.
(9) $2\frac{3}{4}+1\frac{1}{4}$.	" 4.	(19) $3-2\frac{1}{10}$.	" $\frac{9}{10}$.
(10) $4\frac{2}{3}-\frac{7}{3}$.	" $2\frac{1}{3}$.	(20) $3\frac{3}{10}-1\frac{7}{10}$.	" $1\frac{6}{10}$.

Simplify $\frac{9}{17}+\frac{3}{17}+\frac{14}{17}+\frac{11}{17}$. *Ans.* $\frac{37}{17}=2\frac{3}{17}$.

Simplify $8\frac{9}{17}+6\frac{3}{17}+5\frac{14}{17}+\frac{11}{17}$. *Ans.* $19\frac{37}{17}=19+2\frac{3}{17}=21\frac{3}{17}$.

It is generally shorter, however, to add the fractional parts first and carry the wholes; thus, in this last example, the fractions come to $\frac{37}{17}$ or $2\frac{3}{17}$; put down $\frac{3}{17}$ and carry 2 to the wholes. Similarly, in Subtraction we shall first deduct the fractional part, and then the wholes, but if the number of pieces in the subtrahend exceed that in the minuend, we must break up one of the wholes of the minuend, or else add the value of a whole to both terms, as we have done in subtraction of money. Thus:

$$9\frac{5}{8}-7=2\frac{5}{8};$$
$$9\frac{5}{8}-7\frac{3}{8}=2\frac{2}{8};$$
$$9\frac{5}{8}-7\frac{5}{8}=2;$$
$$9-7\frac{5}{8}=8\frac{8}{8}-7\frac{5}{8}=1\frac{3}{8};$$

or, $9 - 7\frac{5}{8} = 9\frac{8}{8} - 8\frac{5}{8} = 1\frac{3}{8}$;

$9\frac{1}{8} - 7\frac{5}{8} = 8\frac{9}{8} - 7\frac{5}{8} = 1\frac{4}{8}$;

or, $9\frac{1}{8} - 7\frac{5}{8} = 9\frac{9}{8} - 8\frac{5}{8} = 1\frac{4}{8}$.

Exercise VI.

(1) $2\frac{1}{2} + 3\frac{1}{2} + 5\frac{1}{2}$.

(2) $8\frac{1}{3} + 7\frac{1}{3} + 15\frac{2}{3} + 1\frac{1}{3}$.

(3) $42\frac{3}{4} + 58\frac{2}{4} + 3\frac{1}{4} + 19\frac{3}{4} + 11\frac{1}{4}$.

(4) $73\frac{4}{5} + 5\frac{2}{5} + 7\frac{1}{5} + 184\frac{3}{5} + 2049\frac{2}{5}$.

(5) $5\frac{13}{17} + 8\frac{16}{17} + 12\frac{11}{17} + 1\frac{9}{17}$.

(6) $3\frac{5}{12} + 52\frac{7}{12} + 87\frac{1}{12} + 943\frac{11}{12}$.

(7) $278\frac{15}{28} - 189$.

(8) $278\frac{15}{28} - 189\frac{15}{28}$.

(9) $278\frac{15}{28} - 189\frac{11}{28}$.

(10) $278 - 189\frac{11}{28}$.

(11) $278\frac{11}{28} - 189\frac{15}{28}$.

(12) $52\frac{3}{10} + 86\frac{1}{10} + 42\frac{9}{10} + 5\frac{7}{10} + 84\frac{2}{10}$.

(13) $584\frac{37}{100} + 129\frac{9}{100} + 43\frac{27}{100} + 3\frac{93}{100} + 4\frac{87}{100} + 1\frac{99}{100} + 6\frac{7}{100}$.

(14) $1\frac{11}{100} + 6\frac{48}{100} + 11\frac{85}{100} + 17\frac{22}{100} + 22\frac{59}{100} + 27\frac{96}{100}$.

(15) $2\frac{43}{1000} + 3\frac{154}{1000} + 4\frac{265}{1000} + 5\frac{376}{1000} + 6\frac{487}{1000} + 7\frac{598}{1000} + 8\frac{709}{1000} + 1\frac{1}{1000}$.

(16) $2012\frac{16}{19} - 789$.

(17) $2012\frac{16}{19} - 789\frac{12}{19}$.

(18) $2012\frac{16}{19} - 789\frac{16}{19}$.

(19) $2012 - 789\frac{12}{19}$.

(20) $2012\frac{12}{19} - 789\frac{18}{19}$.

(21) $34\frac{17}{23} + 18\frac{11}{23} + 49\frac{16}{23} + 519\frac{20}{23} + \frac{21}{23}$.

(22) $4301\frac{10}{11} - 896$.

(23) $4301\frac{10}{11} - 896\frac{4}{11}$.

(24) $4301 - 896\frac{4}{11}$.

(25) $4301\frac{4}{11} - 896\frac{10}{11}$.

(26) $100\frac{4}{11} - (12\frac{7}{11} + 16\frac{8}{11} + 20\frac{9}{11} + 24\frac{10}{11})$.

(27) $(14\frac{8}{17} + 16\frac{8}{17} + 18\frac{13}{17} + 21\frac{1}{17} + 23\frac{6}{17}) - 10\frac{16}{17}$.

(28) $(5\frac{8}{100} + 11\frac{61}{100} + 18\frac{14}{100} + 24\frac{67}{100} + 31\frac{20}{100} + 37\frac{73}{100}) - (1\frac{1}{100} + 7\frac{54}{100} + 14\frac{7}{100} + 20\frac{60}{100} + 27\frac{13}{100} + 33\frac{66}{100})$.

(29) $(538\frac{123}{500} + 169\frac{329}{500}) + (538\frac{123}{500} - 169\frac{329}{500})$.

(30) $(538\frac{123}{500} + 169\frac{329}{500}) - (538\frac{123}{500} - 169\frac{329}{500})$.

(31) $\frac{16}{19}+\frac{16}{19}+\frac{16}{19}+\frac{16}{19}+\frac{16}{19}$.

(32) $7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}+7\frac{13}{25}$.

(33) $8\frac{4}{5}+8\frac{4}{5}+8\frac{4}{5}+8\frac{4}{5}+8\frac{4}{5}$.

§ 8. Multiplication of Fractions by integers.

Exercise E.

(1) How much is 3 times $\frac{4}{9}$? *Ans.* $1\frac{1}{3}$.
(2) „ 5 times $\frac{3}{2}$? „ $7\frac{1}{2}$.
(3) „ 8 times $\frac{3}{2}$? „ 12.
(4) „ 5 times $\frac{2}{3}$? „ $3\frac{1}{3}$.
(5) „ $\frac{1}{3}$ of $\frac{9}{4}$? „ $\frac{3}{4}$.
(6) „ $\frac{1}{3}$ of $4\frac{1}{2}$? „ $1\frac{1}{2}$.
(7) „ $\frac{1}{3}$ of $2\frac{1}{7}$? „ $\frac{5}{7}$.
(8) „ $\frac{1}{5}$ of $2\frac{1}{7}$? „ $\frac{3}{7}$.
(9) How many times can we take $\frac{5}{7}$ out of $2\frac{1}{7}$? *Ans.* 3 times.
(10) „ „ $\frac{3}{7}$ out of $2\frac{1}{7}$? „ 5 times.
(11) „ „ $1\frac{3}{4}$ out of $5\frac{1}{4}$? „ 3 times.
(12) Divide $5\frac{1}{4}$ into 7 equal parts. *Ans.* $\frac{3}{4}$ to each part.
(13) Divide $7\frac{1}{5}$ into 6 equal parts. „ $1\frac{1}{5}$ to each part.
(14) Repeat $\frac{2}{5}$ fourteen times. *Ans.* $5\frac{3}{5}$.
(15) What is the seventh part of $5\frac{3}{5}$? „ $\frac{4}{5}$.
(16) Distribute $4\frac{4}{5}$ cakes among 6 boys.
Ans. $\frac{4}{5}$ of a cake to each boy.

(17) If I walk $2\frac{4}{5}$ miles an hour, how long will it take me to walk $8\frac{2}{5}$ miles? *Ans.* 3 hours.

(18) Take $\frac{4}{7}$ ten times. *Ans.* $5\frac{5}{7}$.
(19) Divide $5\frac{5}{7}$ into 10 equal parts. *Ans.* $\frac{4}{7}$ to each part.
(20) How many times is $\frac{4}{7}$ contained in $5\frac{5}{7}$? *Ans.* 10 times.
(21) Repeat $\frac{5}{8}$ eight times. *Ans.* 5.

(22) I distributed $8\frac{4}{7}$ cwt. of coals among some persons, giving to each $1\frac{5}{7}$ cwt. How many persons were there? *Ans.* 5 persons.

(23) If I spend $\frac{2}{7}$ of a shilling a-day, how much shall I spend in 6 days, and how long will 8*s.* last me?
Ans. $1\frac{5}{7}$*s.* in 6 days; 28 days.

(24) If 1 strip is $1\frac{2}{3}$ yards long, how many strips can I make of 10 yards? *Ans.* 6 strips.

(25) If I sell $\frac{2}{3}$ of a load of hay 8 times, how much do I sell? *Ans.* $5\frac{1}{3}$ loads.

(26) If one cap is made of $\frac{4}{5}$ of a yard, how many yards are wanted for 7 caps? *Ans.* $5\frac{3}{5}$ yards.

(27) If 1 coat requires $2\frac{3}{4}$ yards, how many coats can be cut out of $13\frac{3}{4}$ yards? *Ans.* 5 coats.

(28) How many strips of carpet, each $1\frac{2}{5}$ yards long, can be cut off a remnant $8\frac{2}{5}$ yards long? *Ans.* 6 strips.

(29) How many times will a wheel, $\frac{6}{7}$ of a yard in circumference, turn round in travelling over $12\frac{6}{7}$ yards? *Ans.* 15 times.

(30) How much ground will be travelled over by a wheel, $1\frac{3}{8}$ yards in circumference, after it has made 7 turns? *Ans.* $9\frac{5}{8}$ yards.

(31) How much ground will be travelled over by a wheel, $1\frac{3}{7}$ yards in circumference, after it has made $4\frac{1}{2}$ turns? *Ans.* $6\frac{3}{7}$ yards.

(32) Find the circumference of a wheel which makes 9 turns in travelling over $7\frac{1}{5}$ yards. *Ans.* $\frac{4}{5}$ of a yard.

Let $\frac{1}{\text{F}}$ be 1 farthing, $\frac{1}{\text{HP}} = 1$ halfpenny, $\frac{1}{\text{P}} = 1$ penny, $\frac{1}{\text{S}} = 1$ shilling.

$\frac{1}{\text{F}} \times 2 = \frac{2}{\text{F}} = \frac{1}{\text{HP}}$; again, $\frac{1}{\text{P}} \times 12 = \frac{12}{\text{P}} = \frac{1}{\text{S}}$; similarly, $\frac{7}{\text{P}} \times 12 = \frac{84}{\text{P}} = \frac{7}{\text{S}}$.

In each case here we have two answers, one obtained by alteration of the *number* of the things, and the other by alteration of the *name* of the things, keeping their number the same. In the same way,

$\frac{1}{8} \times 4 = \frac{4}{8} = \frac{1}{2}$ ∵* 1 whole $= \frac{8}{8}$.
$\frac{1}{15} \times 3 = \frac{3}{15} = \frac{1}{5}$ ∵ 1 whole $= \frac{15}{15}$.
$\frac{4}{15} \times 3 = \frac{12}{15} = \frac{4}{5}$, being 4 times as much as $\frac{1}{15} \times 3$.

Comparing the *two* answers to each of these questions, we again perceive that multiplication may be performed either by change of the *number* of the things or of the *name* of the things.

Examine the nature of each of these changes. When the number, i.e. the numerator, is changed, we *multiply*, but when the name, i.e. the denominator, is changed, we *divide*. These two operations cor-

* ∵ = because; ∴ = therefore.

respond to our previous notions of fractions. Thus, if we wish to treble any number of slices of a cake, we may either take three times as many slices, or make each slice three times as large. Suppose each slice to be $\frac{1}{15}$ of a cake; 3 times 4 such slices, or $\frac{4}{15} \times 3$, will be either 12 such slices, i.e. $\frac{12}{15}$, or 4 slices each 3 times as large as $\frac{1}{15}$; but 3 times $\frac{1}{15}$ is $\frac{3}{15} = \frac{1}{5}$, $\therefore$ the 4 slices will be $\frac{4}{5}$; hence $\frac{4}{15} \times 3 = \frac{12}{15}$ or $\frac{4}{5}$. Of these two answers, $\frac{4}{5}$ being the simpler is preferable.

$\frac{7}{16} \times 5 = \frac{35}{16}$. Here, 5 not being a measure of 16, we can at present apply only the first method.

Learn by heart: *To multiply a fraction by an integer, multiply the numerator or divide the denominator. Division is preferable where it can be done without remainder.* Thus:

$\frac{13}{20} \times 2 = \frac{13}{10} = 1\frac{3}{10}$.
$\frac{13}{20} \times 3 = \frac{39}{20} = 1\frac{19}{20}$.
$\frac{13}{20} \times 4 = \frac{13}{5} = 2\frac{3}{5}$.
$\frac{13}{20} \times 5 = \frac{13}{4} = 3\frac{1}{4}$.
$\frac{13}{20} \times 7 = \frac{91}{20} = 4\frac{11}{20}$, &c.

$3\frac{5}{12} \times 2 = 6\frac{5}{6}$.

$3\frac{5}{12} \times 3 = 9\frac{5}{4} = 10\frac{1}{4}$, or, beginning with the fraction, $\frac{5}{12} \times 3 = \frac{5}{4} = 1\frac{1}{4}$, put down $\frac{1}{4}$ and carry 1 whole: $3 \times 3 + 1 = 10$. *Ans.* $10\frac{1}{4}$.

$3\frac{5}{12} \times 4 = 13\frac{2}{3}$.

$3\frac{5}{12} \times 5 = 17\frac{1}{12}$, &c.

Exercise VII.

(1) $\frac{7}{18} \times 2, 3, 4, 5, 6, 7, 8, 9$.

(2) $\frac{11}{60} \times 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 20$.

(3) $\frac{59}{144} \times 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 18, 24, 36, 72$.

(4) $\frac{43}{100} \times 2, 3, 4, 5, 7, 9, 10, 20, 25, 30, 50$.

(5) $\frac{113}{240} \times 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 20, 30, 40, 50, 60, 70, 80, 100, 120$.

(6) $4\frac{1}{2} \times 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(7) $11\frac{11}{30} \times 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(8) $118\frac{419}{1000} \times 2, 4, 5, 7, 11, 20, 24, 25, 100, 500$.

(9) What will 9 men pay for their dinner, if each pays $\frac{3}{4}$ of a crown?

(10) If 1 horse eats $\frac{7}{8}$ of a load, what will 4 horses eat?

(11) If I consume $2\frac{5}{12}$ bushels a-week, how much shall I consume in 7 weeks? Also in 4 weeks?

(12) What will be paid for 6 articles at £$1\frac{11}{18}$ each?

§ 9. Division of Fractions by integers.

Exercise F.

(1) What is $\frac{1}{2}$ of $\frac{1}{2}$? *Ans.* $\frac{1}{4}$.

(2) How many thirds are there in a whole? *Ans.* 3 thirds.

(3) If each third is cut in half, how many of these smaller pieces will there be in the whole? *Ans.* 6 smaller pieces.

(4) What part of the whole would each smaller piece be? *Ans.* $\frac{1}{6}$.

(5) How much then is $\frac{1}{2}$ of $\frac{1}{3}$? „ $\frac{1}{6}$.

(6) How many halves in a whole? *Ans.* 2 halves.

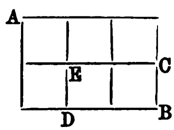
(7) If each half is divided into three equal parts, how many of these smaller pieces will there be in a whole? *Ans.* 6 smaller pieces.

(8) What part of the whole would each piece be? *Ans.* $\frac{1}{6}$.

(9) How much then is $\frac{1}{3}$ of $\frac{1}{2}$? „ $\frac{1}{6}$.

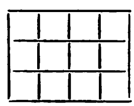
Teacher. Therefore $\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{3}$ of $\frac{1}{2} = \frac{1}{6}$.

Illustration :



AB is the whole, AC the half, AD the third, AE either half of the third, or one-third of the half, or one-sixth of the whole.

Similarly $\frac{1}{3}$ of $\frac{1}{4} = \frac{1}{4}$ of $\frac{1}{3} = \frac{1}{12}$:



(10) $\frac{1}{4}$ of $\frac{1}{5}$. *Ans.* $\frac{1}{20}$.

(11) $\frac{1}{5}$ of $\frac{1}{4}$. „ $\frac{1}{20}$.

(12) $\frac{1}{3}$ of $\frac{1}{8}$. „ $\frac{1}{24}$.

(13) $\frac{1}{8}$ of $\frac{1}{3}$. „ $\frac{1}{24}$.

(14) $\frac{1}{6}$ of $\frac{1}{10}$. „ $\frac{1}{60}$.

(15) $\frac{1}{2} \div 2$. „ $\frac{1}{4}$.

(16) $\frac{1}{2} \div 3$. *Ans.* $\frac{1}{6}$.

(17) $\frac{1}{3} \div 2$. „ $\frac{1}{6}$.

(18) $\frac{1}{3} + 4$. „ $\frac{1}{12}$.

(19) $\frac{1}{5} \div 4$. „ $\frac{1}{20}$.

(20) $\frac{1}{10} \div 6$. „ $\frac{1}{60}$.

(21) $\frac{1}{8}$ of $\frac{1}{9}$. „ $\frac{1}{72}$.

(22) $\frac{1}{9}$ of $\frac{1}{8}$. *Ans.* $\frac{1}{72}$.
(23) $\frac{1}{9} \div 8$. " $\frac{1}{72}$.
(24) $\frac{1}{8} \div 9$. " $\frac{1}{72}$.
(25) $\frac{1}{7}$ of $\frac{1}{5}$. " $\frac{1}{35}$.
(26) $\frac{1}{7} \div 5$. " $\frac{1}{35}$.
(27) $\frac{1}{5}$ of $\frac{1}{7}$. " $\frac{1}{35}$.
(28) $\frac{1}{5} \div 7$. " $\frac{1}{35}$.
(29) $\frac{1}{5}$ of $\frac{1}{9}$. " $\frac{1}{45}$.
(30) $\frac{1}{9}$ of $\frac{1}{5}$. " $\frac{1}{45}$.
(31) $\frac{1}{8}$ of $\frac{1}{7}$. " $\frac{1}{56}$.
(32) $\frac{1}{7}$ of $\frac{1}{8}$. " $\frac{1}{56}$.
(33) $\frac{1}{4}$ of $\frac{1}{7}$. " $\frac{1}{28}$.
(34) $\frac{1}{7}$ of $\frac{1}{4}$. " $\frac{1}{28}$.
(35) $\frac{1}{9}$ of $\frac{1}{10}$. " $\frac{1}{90}$.
(36) $\frac{1}{10}$ of $\frac{1}{9}$. " $\frac{1}{90}$.
(37) $\frac{1}{5}$ of $\frac{1}{3}$. " $\frac{1}{15}$.
(38) $\frac{1}{3}$ of $\frac{1}{5}$. " $\frac{1}{15}$.
(39) $\frac{1}{10}$ of $\frac{1}{10}$. " $\frac{1}{100}$.
(40) $\frac{1}{8}$ of $\frac{1}{8}$. " $\frac{1}{64}$.
(41) $\frac{1}{9}$ of $\frac{1}{9}$. " $\frac{1}{81}$.
(42) $\frac{1}{5}$ of $\frac{1}{5}$. " $\frac{1}{25}$.
(43) $\frac{1}{10}$ of $\frac{1}{11}$. *Ans.* $\frac{1}{110}$.
(44) $\frac{1}{11}$ of $\frac{1}{10}$. " $\frac{1}{110}$.
(45) $\frac{1}{11}$ of $\frac{1}{11}$. " $\frac{1}{121}$.
(46) $\frac{1}{11}$ of $\frac{1}{12}$. " $\frac{1}{132}$.
(47) $\frac{1}{12}$ of $\frac{1}{11}$. " $\frac{1}{132}$.
(48) $\frac{1}{5} \div 9$. " $\frac{1}{45}$.
(49) $\frac{1}{9} \div 5$. " $\frac{1}{45}$.
(50) $\frac{1}{8} \div 7$. " $\frac{1}{56}$.
(51) $\frac{1}{7} \div 8$. " $\frac{1}{56}$.
(52) $\frac{1}{7} \div 4$. " $\frac{1}{28}$.
(53) $\frac{1}{4} \div 7$. " $\frac{1}{28}$.
(54) $\frac{1}{8} \div 3$. " $\frac{1}{24}$.
(55) $\frac{1}{3} \div 8$. " $\frac{1}{24}$.
(56) $\frac{1}{11} \div 12$. " $\frac{1}{132}$.
(57) $\frac{1}{12} \div 11$. " $\frac{1}{132}$.
(58) $\frac{1}{5} \div 8$. " $\frac{1}{40}$.
(59) $\frac{1}{10} \div 4$. " $\frac{1}{40}$.
(60) $\frac{1}{20} \div 2$. " $\frac{1}{40}$.
(61) $\frac{1}{2} \div 20$. " $\frac{1}{40}$.
(62) $\frac{1}{8} \div 5$. " $\frac{1}{40}$.
(63) $\frac{1}{10} \div 10$. " $\frac{1}{100}$.

$\frac{6}{\text{P}} \div 2$ means that 6 pence are to be distributed into 2 equal parts. *Ans.* $\frac{3}{\text{P}}$ to each part.

$\frac{1}{\text{P}} \div 2 = \frac{1}{\text{HP}}$; similarly, $\frac{5}{\text{P}} \div 2 = \frac{5}{\text{HP}}$; and $\frac{6}{\text{P}} \div 2 = \frac{6}{\text{HP}} = \frac{3}{\text{P}}$. In this last case, we have *two* answers, one obtained by alteration of the *number*, and the other by alteration of the *name*. In the same way, $\frac{8}{7} \div 2 = \frac{4}{7}$ to each part. $\frac{1}{7} \div 2 = ?$ Here we cannot divide the *number;* can we solve the question by altering the *name?* We have to divide a seventh into two equal parts, i.e. to find the half of $\frac{1}{7}$. *Ans.* $\frac{1}{14}$ to each part. Similarly, $\frac{5}{7} \div 2 = \frac{5}{14}$ to each part; also $\frac{8}{7} \div 2 = \frac{8}{14}$; but $\frac{8}{7} \div 2 = \frac{4}{7}$, therefore we obtain two answers, one by *dividing* the *numerator*, the other by *multiplying* the *denominator*. As before, these two operations correspond to our previous notions of fractions. Thus if we wish to distribute any number of slices of a cake among three persons, we may either distribute the slices, or break up each slice into three equal parts. Suppose each

slice to be $\frac{1}{7}$ of a whole cake; in dividing 6 such slices among 3 persons, we may either give 2 slices to each, i.e. $\frac{2}{7}$, or we may break up each seventh into 3 parts, and give to each person 6 such parts, of which each is $\frac{1}{3}$ of $\frac{1}{7}$, or $\frac{1}{21}$, therefore each person will have $\frac{6}{21}$. Of these two answers, $\frac{2}{7}$ being the simpler is preferable.

$\frac{5}{7} \div 3$. Here, 3 not being a measure of 5, we can only apply the method of breaking up the pieces. *Ans.* $\frac{5}{21}$.

Learn by heart: *To divide a fraction by an integer, either divide the numerator or multiply the denominator. Division is preferable where it can be done without remainder.*

$\frac{8}{9} \div 2 = \frac{4}{9}$ to each part.
$\frac{8}{9} \div 3 = \frac{8}{27}$ „
$\frac{8}{9} \div 4 = \frac{2}{9}$ „
$\frac{8}{9} \div 5 = \frac{8}{45}$ „
$\frac{8}{9} \div 8 = \frac{1}{9}$ „
$1\frac{5}{7} \div 2 = \frac{12}{7} \div 2 = \frac{6}{7}$ to each part.
$1\frac{5}{7} \div 3 = \frac{12}{7} \div 3 = \frac{4}{7}$ „
$1\frac{5}{7} \div 4 = \frac{12}{7} \div 4 = \frac{3}{7}$ „
$1\frac{5}{7} \div 5 = \frac{12}{7} \div 5 = \frac{12}{35}$ „
$1\frac{5}{7} \div 6 = \frac{12}{7} \div 6 = \frac{2}{7}$ „
$1\frac{5}{7} \div 7 = \frac{12}{7} \div 7 = \frac{12}{49}$ „
$1\frac{5}{7} \div 12 = \frac{12}{7} \div 12 = \frac{1}{7}$ to each part.
$14\frac{2}{7} \div 2 = 7\frac{1}{7}$ to each part.
$14\frac{2}{7} \div 3 = 4\frac{16}{21}$ „

Working: $3)\underline{14\frac{2}{7}}$
4 wholes and $2\frac{2}{7}$ or $\frac{16}{7}$ over; $\frac{16}{7} \div 3 = \frac{16}{21}$.

$14\frac{2}{7} \div 4 = 3\frac{4}{7}$ to each part.

Working: $4)\underline{14\frac{2}{7}}$
3 and $2\frac{2}{7}$ or $\frac{16}{7}$ over; $\frac{16}{7} \div 4 = \frac{4}{7}$.

$14\frac{2}{7} \div 5 = 2\frac{6}{7}$ to each part.

Working: $5)\underline{14\frac{2}{7}}$
2 and $4\frac{2}{7}$ or $\frac{30}{7}$ over; $\frac{30}{7} \div 5 = \frac{6}{7}$.

$14\frac{2}{7} \div 6 = 2\frac{16}{42}$ to each part.

Working: $6)\underline{14\frac{2}{7}}$
2 and $2\frac{2}{7}$ or $\frac{16}{7}$ over; $\frac{16}{7} \div 6 = \frac{16}{42}$.

Exercise VIII.

(1) $\frac{15}{16} \div 2, 3, 4, 5, 6, 7, 8.$

(2) $\frac{20}{21} \div 2, 3, 4, 5, 6, 7, 10.$

(3) $\frac{100}{107} \div 2, 3, 4, 5, 6, 10, 20, 100.$

(4) $5\frac{1}{4} \div 2, 3, 4, 5, 7, 10.$

(5) $38\frac{2}{5} \div 2, 3, 4, 5, 6, 7, 8, 12, 20.$

(6) $1423\frac{1}{17} \div 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$

§ 10. The two rules for multiplication and division of fractions by integers can be summarized thus: Either perform on the numerator the operation indicated by the sign, or on the denominator the opposite of what the sign indicates. In each case division is preferable to multiplication where it can be done without remainder.

Exercise IX.

(1) $\frac{144}{175} \times 2, 3, 4, 5, 6, 7, 8, 25, 35, 175.$
$\div 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 72.$

(2) $8\frac{74}{77} \times 2, 3, 4, 7, 11, 12, 77.$
$\div 2, 3, 4, 5, 6, 7, 10, 23, 230.$

(3) Find $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ of $\frac{12}{13}$.

(4) Find $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ $\frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ of $860\frac{8}{11}$.

(5) A Hanoverian mile is $4\frac{13}{22}$ English miles nearly. Find the length in English miles of 11 Hanoverian miles.

(6) Find $\frac{2}{3}, \frac{3}{4}, \frac{3}{5}, \frac{5}{8}, \frac{7}{9}, \frac{7}{10}$ of $2522\frac{31}{72}$.

(7) How long will $13\frac{2}{7}$ pieces be, if 3 are $8\frac{2}{5}$ yards in length?

(8) Distribute $9\frac{1}{3}$ cwt. of potatoes among 7 families of 5 persons each. How much will be given to each family, and how much to each person?

(9) If 1 person consumes $\frac{4}{15}$ of a lb. a-day, how much will a family of 5 persons consume in a week?

(10) Find the average of the following lengths: $1\frac{4}{11}$ yds., $1\frac{6}{11}$ yds., $1\frac{7}{11}$ yds., $\frac{10}{11}$ yds., $1\frac{9}{11}$ yds.

(11) I empty into a vat 2 vessels of $4\frac{5}{7}$ gallons each, 3 vessels of $2\frac{6}{7}$ gals. each, 7 vessels of $1\frac{4}{7}$ gals. each, 5 vessels of $4\frac{1}{7}$ gals. each. Distribute the contents of the vat into 2 equal vessels; also into 3, 4, 5, 7, 12, and 29 equal vessels.

(12) If I walk $3\frac{5}{8}$ miles an hour, how far shall I walk in 2, 3, 4, 5, 6 hours?

§ 11. $\frac{1}{8} \times 8 = 1$ whole; therefore $\frac{3}{8} \times 8 = 3$ wholes, $\frac{7}{8} \times 8 = 7$ wholes. Similarly $\frac{1}{11} \times 11 = 1$ whole, $\frac{6}{11} \times 11 = 6$ wholes, &c. Hence:

Learn by heart: *Any fraction multiplied by its denominator gives for answer its numerator as a whole number.*

Exercise X.

(1) $\frac{5}{13} \times 13$.
(2) $\frac{19}{23} \times 23$.
(3) $\frac{2485}{9711} \times 9711$.
(4) $6\frac{4}{5} \times 5$.
(5) $10\frac{7}{19} \times 19$.
(6) $463\frac{61}{62} \times 62$.
(7) $8\frac{4}{7} \times 7, 14, 21$.
(8) $10\frac{5}{12} \times 24, 36, 60$.

Exercise G.

§ 12. (1) How many times are $\frac{3}{4}$ contained in $5\frac{1}{4}$? *Ans.* 7 times.

(2) If one cap requires $1\frac{3}{4}$ yards, how many can be made out of $5\frac{1}{4}$ yards? *Ans.* 3 caps.

(3) A distance of $5\frac{5}{11}$ miles is divided into lengths of $\frac{6}{11}$ of a mile each. How many such lengths are there? *Ans.* 10 lengths.

(4) By what number must $\frac{4}{9}$ be multiplied to give $4\frac{8}{9}$? *Ans.* By 11.

(5) How many times is $\frac{3}{8}$ contained in 9 wholes? *Ans.* 24 times.

(6) If one dinner cost $\frac{3}{5}$ of half-a-crown, how many dinners will 3 half-crowns pay for? *Ans.* 5 dinners.

(7) Interpret $8\frac{2}{5} \div \frac{2}{5}$.

Ans. How many times are $\frac{2}{5}$ contained in $8\frac{2}{5}$.

(8) $8\frac{2}{5} \div \frac{2}{5}$. *Ans.* 21 times.

(9) Interpret $4\frac{2}{7} \div \frac{6}{7}$ and $4\frac{2}{7} \div 6$.

Ans. The first means, How many times are $\frac{6}{7}$ contained in $4\frac{2}{7}$; the second means, Distribute $4\frac{2}{7}$ into 6 equal parts.

(10) What are the answers then?

Ans. 5 times; and $\frac{5}{7}$ to each part.

(11) If a man spends $\frac{2}{5}$ of his daily wages, how many days will it take him to accumulate 4 days' wages? *Ans.* 10 days.

(12) $(1\frac{3}{7} + \frac{6}{7}) \div (1\frac{3}{7} - \frac{6}{7})$. *Ans.* 4 times.

Exercise XI.

(1) $76\frac{12}{23} \div \frac{1}{23}, \frac{2}{23}, \frac{4}{23}, \frac{5}{23}, \frac{8}{23}, \frac{11}{23}, \frac{16}{23}$.

(2) $76\frac{12}{23} \div 3\frac{11}{23}, 4\frac{18}{23}, 7\frac{15}{23}, 19\frac{3}{23}$.

(3) If a wheel is $2\frac{6}{7}$ feet in circumference, how many turns will it make in travelling over $17\frac{1}{7}$ feet?

(4) How many times is $\frac{5}{17}$ contained in $4\frac{12}{17}$?

(5) To how many people can I give $\frac{11}{21}$ of a load each out of $5\frac{5}{21}$ loads?

Exercise XII.

(1) If I consume $2\frac{5}{8}$ bushels in one week, how much shall I consume in 3 days; also in a quarter of a day?

(2) Divide $3417\frac{4}{9}$ by 45.

(3) If I spend alternately £$\frac{2}{7}$ and £$\frac{3}{7}$ in a day, how much shall I spend in 14 days, and how long will £$8\frac{4}{7}$ last me?

(4) Find the value of $\frac{11}{14}$ of a guinea + $\frac{3}{8}$ of a shilling — $\frac{5}{6}$ of 9*d.*

(5) Find the value of $\frac{13}{27}$ of £209. 11*s.* $2\frac{1}{4}$*d.*

(6) $2\frac{26}{35} \times 2, 3, 4, 5, 6, 7, 8, 9, 10, 35$.

(7) $2\frac{26}{35} \div 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(8) $2\frac{26}{35} \div \frac{1}{35}, \frac{8}{35}, \frac{24}{35}, 1\frac{13}{35}$.

(9) If $\frac{5}{7}$ of a ship cost £3259. 11*s.* 8*d.*, what will the whole cost? Also what will $\frac{3}{7}$ cost?

(10) If a certain number of trusses of hay were accurately distributed among 359 horses, each would have $2\frac{82}{359}$ trusses. How many trusses were to be divided?

(11) Simplify $(9\frac{10}{11} + 11\frac{9}{11} + 13\frac{8}{11} + 15\frac{7}{11} + 17\frac{6}{11} + 19\frac{5}{11})$ — $(8\frac{7}{11} + 7\frac{5}{11} + 6\frac{9}{11} + 6\frac{2}{11} + 5\frac{6}{11} + 4\frac{10}{11})$.

(12) How many times is the difference between $\frac{1}{5}$ of $1\frac{7}{28}$ and $\frac{1}{7}$ of $1\frac{1}{4}$ contained in the sum of 3 times $10\frac{11}{28}$ and 5 times $9\frac{3}{140}$?

(13) Find the length of 7 pieces, if each is $12\frac{5}{9}$ yards long.

(14) Find the length of $5\frac{3}{4}$ pieces, if each is 60 yards long.

(15) Find the length of $147\frac{6}{7}$ pieces, if 9 pieces are $113\frac{2}{5}$ yards long.

CHAPTER II.

INTERCONVERSION OF DENOMINATORS.

§ 1. We have seen that by multiplying the numerator, the value of the fraction is multiplied; by multiplying the denominator, the value of the fraction is divided. Hence:

By multiplying both numerator and denominator *by the same number*, the value of the fraction is not altered. Thus $\frac{5}{7}=\frac{5\times 6}{7\times 6}=\frac{30}{42}$. We have, in fact, six times as many pieces as before, but each piece is one-sixth of the original size.

By dividing the numerator, the value of the fraction is divided; by dividing the denominator, the value of the fraction is multiplied. Hence:

By dividing both numerator and denominator by the same number, the value of the fraction is not altered. Thus: $\frac{30}{42}=\frac{30\div 6}{42\div 6}=\frac{5}{7}$. We have, in fact, one-sixth of the original number of pieces, but each piece is six times as great as before.

The numerator and denominator are called the *terms* of the fraction. By dividing numerator and denominator of a fraction by the same number, the fraction is reduced to lower terms, and when the terms are prime to each other, the fraction is at its *lowest terms*.

Reduce $\frac{61776}{80784}$ to lowest terms.

Applying the tests of Part I. Ch. XI. § 8, we instantly find that both terms are divisible by 4, $\therefore \frac{61776}{80784}=\frac{15444}{20196}$. This fraction is again reducible by 4, $\therefore \frac{15444}{20196}=\frac{3861}{5049}$. We have now got rid of all *even* common measures, and applying the test for 3 or 9, we find the fraction reducible by 9, $\therefore \frac{3861}{5049}=\frac{429}{561}$. Repeating this test, we find it reducible by 3 and not by 9, $\therefore \frac{429}{561}=\frac{143}{187}$. We have now got rid of all common measures which are multiples of 3. Applying the test for 11, we find the fraction reducible by 11, $\therefore \frac{143}{187}=\frac{13}{17}$. 13 and 17 being prime to each other, the fraction is reduced to its lowest terms.

	4)	4)	9)	3)	11)	
Mod. op.:	61776	15444	3861	429	143	13
	80784	20196	5049	561	187	17

Reduce $\frac{4189}{4307}$ to lowest terms. Here none of the tests above referred to reveal a common measure. Find G.C.M. of the terms. It is 59.

$$\begin{array}{r|l} \multicolumn{2}{c}{1,\ 71} \\ \hline 4189 & 4307 \\ 59 & 118 \\ \cline{2-2} & 59 \end{array} \qquad \therefore \qquad 59)\ \frac{4189}{4307}\Big|\frac{71}{73}$$

Reduce $\frac{1287}{4760}$ to lowest terms. The tests give us no common measure, and G.C.M. is found to be 1. Therefore the fraction is already at its lowest terms.

$$\frac{1287}{117}\Big|\frac{4760}{119}$$

Reduce $\frac{7850304}{13083840}$ to lowest terms.

$$4)\ \frac{7850304}{13083840}\Big|\ 4)\ \frac{1962576}{3270960}\Big|\ 4)\ \frac{490644}{817740}\Big|\ 9)\ \frac{122661}{204435}\Big|\ 11)\ \frac{13629}{22715}\Big|\ 413)\ \frac{1239}{2065}\Big|\ \frac{3}{5}$$

Here the first five divisors were found by inspection; the last by the process for G.C.M.

$$\frac{1239}{413}\Big|\frac{2065}{413}$$

To test the accuracy of the result, divide the original numerator by the last numerator, and the original denominator by the last denominator. The two quotients should obviously be the same.

$$\begin{array}{l} 3)\underline{7850304} \\ \quad 2616768 \end{array} \qquad \begin{array}{l} 5)\underline{13083840} \\ \quad 2616768 \end{array}$$

EXERCISE XIII.

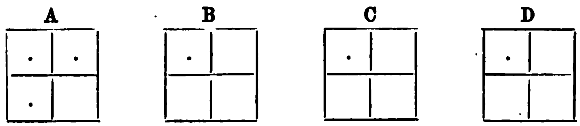
Reduce to lowest terms:

(1) $\frac{20}{25}$.	(14) $\frac{960}{1000}$.	(27) $\frac{44828}{61087}$.	(40) $\frac{6}{9}$.
(2) $\frac{36}{48}$.	(15) $\frac{85}{100}$.	(28) $\frac{339}{1243}$.	(41) $\frac{68}{99}$.
(3) $\frac{42}{77}$.	(16) $\frac{875}{10000}$.	(29) $\frac{1177}{2675}$.	(42) $\frac{74}{999}$.
(4) $\frac{18}{27}$.	(17) $\frac{4375}{10000}$.	(30) $\frac{11445}{15369}$.	(43) $\frac{27}{999}$.
(5) $\frac{28}{112}$.	(18) $\frac{224}{1000}$.	(31) $\frac{85359}{94128}$.	(44) $\frac{630}{9999}$.
(6) $\frac{56}{84}$.	(19) $\frac{2640}{2970}$.	(32) $\frac{85359}{86128}$.	(45) $\frac{143}{9999}$.
(7) $\frac{98}{112}$.	(20) $\frac{324}{1092}$.	(33) $\frac{6171}{6732}$.	(46) $\frac{9090}{9999}$.
(8) $\frac{75}{100}$.	(21) $\frac{924}{1092}$.	(34) $\frac{14141}{16289}$.	(47) $\frac{720}{99999}$.
(9) $\frac{1300}{1700}$.	(22) $\frac{6732}{9108}$.	(35) $\frac{881496}{104768}$.	(48) $\frac{3280}{99999}$.
(10) $\frac{625}{1200}$.	(23) $\frac{3872}{92807}$.	(36) $\frac{2760}{4485}$.	(49) $\frac{29810}{99999}$.
(11) $\frac{66}{121}$.	(24) $\frac{6840}{27360}$.	(37) $\frac{5760}{7000}$.	(50) $\frac{6216}{999999}$.
(12) $\frac{143}{176}$.	(25) $\frac{78473}{94653}$.	(38) $\frac{2205}{2240}$.	(51) $\frac{65065}{999999}$.
(13) $\frac{375}{1000}$.	(26) $\frac{17596}{26145}$.	(39) $\frac{14028}{28392}$.	(52) $\frac{8925845}{10881585}$.

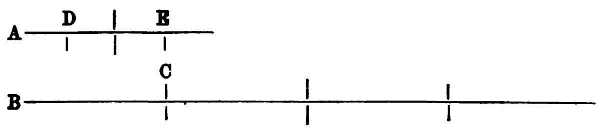
(53) If a cake is divided equally among a school of 75 pupils, 30 being boys, what part of the whole cake do the boys get, and what the girls?

(54) The property of a mining company is divided into 6000 shares. A holds 625 shares, B 800 shares, and C 900 shares. What part of the whole mine does each possess, and what part do they hold jointly?

§ 2. We have hitherto attached to the symbol $\frac{3}{4}$ the interpretation that *one* whole is divided into four equal parts, of which *three* are taken. If, however, *three* wholes are divided into four equal parts and *one* of these parts is taken, we obtain a piece of equal size, for a quarter of three must be three times as great as a quarter of one. Thus the same piece is obtained by taking three quarters of the square A, or one quarter of the three squares B, C and D.



Similarly, if the line A is one inch, and B three inches, one quarter of B, viz. BC, is three times as long as AD the quarter of A, and is therefore as long as AE.



Learn by heart: $\frac{1}{4}$ of $3 = \frac{3}{4}$ of 1. Thus:

$\frac{1}{4}$ of £3 = $\frac{1}{4}$ of 60*s*. = 15*s*.
$\frac{3}{4}$ of £1 = 3 × 5*s* = 15*s*.
$\frac{1}{4}$ of 3*s*. = $\frac{1}{4}$ of 36*d*. = 9*d*.
$\frac{3}{4}$ of 1*s*. = 3 × 3*d*. = 9*d*.

$\frac{3}{4}$ of £5. 11*s*. 8*d*. = $\frac{1}{4}$ of 3 × £5. 11*s*. 8*d*.

	£.	*s*.	*d*.
4)	5	11	8
	1	7	11
			3
	£4	3	9

	£.	*s*.	*d*.
	5	11	8
			3
4)	16	15	0
	£4	3	9

Similarly, $\frac{1}{7}$ of $5 = \frac{5}{7}$ of 1. Find $\frac{5}{7}$ of £8. 18*s*. $9\frac{1}{2}d$. in two ways.

$$\begin{array}{r} 7)\underline{8 \;\; 18 \;\; 9\frac{1}{2}} \\ 1 \;\; 5 \;\; 6\frac{1}{2} \\ \underline{5} \\ £6 \;\; 7 \;\; 8\frac{1}{2} \end{array} \qquad\qquad \begin{array}{r} £8 \;\; 18 \;\; 9\frac{1}{2} \\ \underline{5} \\ 7)\underline{44 \;\; 13 \;\; 11\frac{1}{2}} \\ £6 \;\; 7 \;\; 8\frac{1}{2} \end{array}$$

EXERCISE XIV.

Find in two ways:

(1) $\frac{6}{7}$ of £940. 7*s*. 3*d*.

(2) $\frac{10}{11}$ of £1888. 13*s*. 1*d*.

(3) $\frac{5}{12}$ of £2060. 1*s*. 3*d*.

(4) $\frac{5}{6}$ of £21. 19*s*. 3*d*.

(5) $\frac{4}{11}$ of £25. 19*s*. $11\frac{3}{4}d$.

(6) $\frac{8}{9}$ of £17. 4*s*. $7\frac{1}{2}d$.

EXERCISE H.

(1) What is $\frac{1}{8}$ of 3? *Ans.* $\frac{3}{8}$.

(2) Divide 3 into 8 equal parts. *Ans.* $\frac{3}{8}$ to each part.

(3) If I spend 6*s*. in 7 days, what fraction of 1*s*. do I spend per day? *Ans.* $\frac{6}{7}$ of 1*s*.

(4) Distribute 7 yards into 11 equal parts. *Ans.* $\frac{7}{11}$ of a yard to each part.

(5) Divide 11 yards into 7 equal parts. *Ans.* $1\frac{4}{7}$ yards to each part.

(6) If 2 oranges are divided among 3 children, what will each child get? *Ans.* $\frac{2}{3}$ of an orange.

(7) Divide 4 cakes among 7 children. *Ans.* $\frac{4}{7}$ of a cake to each.

(8) Divide 7 cakes among 4 children. *Ans.* $\frac{7}{4}$ or $1\frac{3}{4}$ cakes to each.

(9) $\frac{1}{6}$ of 9. *Ans.* $1\frac{1}{2}$.

(10) $\frac{1}{9}$ of 6. „ $\frac{6}{9}$ or $\frac{2}{3}$.

(11) $\frac{1}{8}$ of 15. „ $1\frac{7}{8}$.

(12) $\frac{1}{15}$ of 8. „ $\frac{8}{15}$.

(13) $\frac{1}{20}$ of 30. „ $1\frac{1}{2}$.

(14) $\frac{1}{30}$ of 20. „ $\frac{2}{3}$.

(15) $\frac{1}{30}$ of 25. „ $\frac{5}{6}$.

(16) $\frac{1}{25}$ of 30. *Ans.* $1\frac{1}{5}$.

(17) $8 \div 9$. „ $\frac{8}{9}$.

(18) $7 \div 10$. „ $\frac{7}{10}$.

(19) $10 \div 7$. „ $1\frac{3}{7}$.

(20) $15 \div 4$. „ $\frac{15}{4}$ or $3\frac{3}{4}$.

(21) $67 \div 12$. „ $5\frac{7}{12}$.

(22) $12 \div 67$. „ $\frac{12}{67}$.

From this it follows that every fraction indicates a division, and every division a fraction, the dividend being the numerator, the divisor the denominator, and the quotient the value of the fraction. It is well to remember that

Dividend,	Divisor,	Quotient,
Numerator,	Denominator,	Fraction,

are two sets of names with identical meanings.

EXERCISE XV.

(1) $\frac{1}{74}$ of 111.
(2) $\frac{1}{111}$ of 74.
(3) Divide 60 things amongst 42 persons.
(4) „ 42 things amongst 60 persons.
(5) „ 108 things amongst 144 persons.
(6) „ 144 things into 108 equal parts.
(7) „ 520 things among 195 persons.
(8) „ 195 yards into 520 equal lengths.

§ 3. Distribute 75419 into 156 equal parts.

156) 75419 (483 to each part.
1301
539
71 over.

We can now dispose of this remainder 71. It is to be distributed into 156 equal parts, each part will therefore be $\frac{1}{156}$ of 71 $= \frac{71}{156}$. The complete answer, therefore, is $483\frac{71}{156}$ to each part.

15372 ÷ 792.

792) 15372 ($19\frac{9}{22}$
7452
$\frac{324}{792} \,\Big|\, \frac{81}{198} \,\Big|\, \frac{9}{22}$ *Ans.* $19\frac{9}{22}$ to each part.

16043 ÷ 72.

8) 16043
9) $2005\frac{3}{8}$
$222\frac{59}{72}$. *Ans.* $222\frac{59}{72}$ to each part.

N.B. On comparing this process with that given in Part I. Ch. IX. § 15, et seq., it will be seen that they are identical.

EXERCISE XVI.

(1) 17429÷387
(2) 150768÷1224
(3) 150768÷132
(4) 111114÷41
(5) 86354÷45
(6) 1000000÷1625
(7) 195÷610
(8) 26813÷73
(9) 26813÷28
(10) 3465÷5355
(11) 155554÷2439
(12) 73043÷42
(13) 4016093÷1517
(14) 18467÷2400
(15) 77900÷1685
(16) 12219÷165
(17) 438÷2100
(18) 2100÷438.

§ 4. £517. 6*s.* 11*d.* ÷ 7.

7) 517 6 11
£73 18 $1\frac{4}{7}$

When there are no farthings in the dividend, the pence over should not be reduced to farthings, but at once expressed as a fraction of a penny.

Find in two ways $\frac{5}{9}$ of £417. 6*s.* 5*d.*

$$\begin{array}{r}
9)\underline{417\quad 6\quad 5} \\
46\quad 7\quad 4\frac{5}{9} \\
\underline{5} \\
£231\quad 16\quad 10\frac{7}{9}
\end{array}
\qquad\qquad
\begin{array}{r}
417\quad 6\quad 5 \\
\underline{5} \\
9)\underline{2086\quad 12\quad 1} \\
£231\quad 16\quad 10\frac{7}{9}
\end{array}$$

£53. 8*s.* $9\frac{1}{4}$*d.* $\div$ 7.

$$\begin{array}{r}
7)\underline{53\quad 8\quad 9\frac{1}{4}} \\
£7\quad 12\quad 8\frac{5}{28}
\end{array}$$

Here the $1\frac{1}{4}$*d.* over must be made into farthings, i.e. fourths, $\frac{5}{4} \div 7 = \frac{5}{28}$.

£103. 8*s.* $10\frac{2}{5}$*d.* $\div$ 6.

$$\begin{array}{r}
6)\underline{103\quad 8\quad 10\frac{2}{5}} \\
£17\quad 4\quad 9\frac{22}{30}\Big|\frac{11}{15}
\end{array}$$

Here the $4\frac{2}{5}$ over must be made into fifths of a penny. $\frac{22}{5} \div 6 = \frac{22}{30} = \frac{11}{15}$ of a penny.

£23281. 15*s.* $7\frac{1}{2}$*d.* $\div$ 615.

$$\begin{array}{l}
615)23281\quad 15\quad 7\frac{1}{2}(37\quad 17\quad 1\frac{47}{82} \\
\quad\;\; 4831 \\
\quad\;\; \underline{526} \\
\quad\;\; 1053 \\
\quad\;\;\; 4385 \\
\quad\;\;\; \underline{\;\;80} \\
\quad\;\;\; 967 \\
\quad\;\;\; \underline{352} \\
\quad\;\;\; \frac{705}{2} \div 615 = \frac{705}{1230}\Big|\frac{141}{246}\Big|\frac{47}{82}
\end{array}$$

EXERCISE XVII.

(1) £4617. 13*s.* 5*d.* $\div$ 2, 3, 4, 5, 6, 7, 8, 9, 10.

(2) £4617. 13*s.* 5*d.* $\div$ 17, 151, 367, 5928.

(3) £4617. 13*s.* 5*d.* $\div$ 42, 63, 84, 108.

(4) £3815. 6*s.* $2\frac{3}{4}$*d.* $\div$ 2, 3, 4, 5, 6, 7, 8, 9, 10.

(5) £3815. 6*s.* $2\frac{3}{4}$*d.* $\div$ 17, 151, 367, 5928.

(6) £3815. 6*s.* $2\frac{3}{4}$*d.* $\div$ 42, 63, 84, 108.

(7) Find in two ways: $\frac{5}{9}$ of £23. 14*s.* 2*d.*, £23. 14*s.* $2\frac{1}{2}$*d.*, £23. 14*s.* $2\frac{3}{4}$*d.*, £23. 14*s.* $2\frac{5}{7}$*d.*

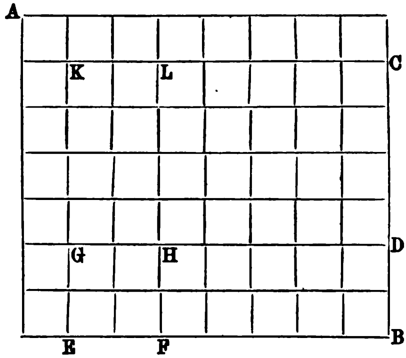
(8) Find $\frac{6}{11}$ of £897. 12*s.* 10*d.*, £897. 12*s.* $10\frac{1}{4}$*d.*, £897. 12*s.* $10\frac{3}{8}$*d.*

(9) Divide $3417\frac{4}{5}$ by 42 in three different ways.

(10) Find the value of £$\frac{6}{13}$, $\frac{7}{8}$ of a guinea, $\frac{3}{5}$ of 1*s.*

§ 5. Complex Fractions.

It is required to find $\frac{5}{7}$ of $\frac{3}{8}$. $\frac{1}{7}$ of $\frac{1}{8}=\frac{1}{56}$ (Part II. Ch. I. § 9), $\therefore$ $\frac{1}{7}$ of $\frac{3}{8}=\frac{3}{56}$, and $\therefore$ $\frac{5}{7}$ of $\frac{3}{8}=5\times\frac{3}{56}=\frac{15}{56}$. Similarly, $\frac{3}{8}$ of $\frac{5}{7}=3\times\frac{1}{8}$ of $\frac{5}{7}=3\times\frac{5}{56}=\frac{15}{56}$. Hence $\frac{5}{7}$ of $\frac{3}{8}=\frac{3}{8}$ of $\frac{5}{7}=\frac{15}{56}$.



From A to C is $\frac{1}{7}$ of the whole AB, $\therefore$ AD is $\frac{5}{7}$ of AB.

AG is $\frac{1}{8}$ of AD or $\frac{1}{8}$ of $\frac{5}{7}$, $\therefore$ AH is $\frac{3}{8}$ of $\frac{5}{7}$, or 15 times the piece AK, i.e. $\frac{15}{56}$ of AB.

Again, AE is $\frac{1}{8}$ of AB, $\therefore$ AF is $\frac{3}{8}$ of AB.

AL is $\frac{1}{7}$ of AF or $\frac{1}{7}$ of $\frac{3}{8}$, $\therefore$ AH is $\frac{5}{7}$ of $\frac{3}{8}=\frac{15}{56}$ as before.

Learn by heart: *To simplify a complex fraction, multiply the two numerators together for the new numerator, and the two denominators for the new denominator.*

Exercise K.

(1) $\frac{1}{4}$ of 3.	*Ans.* $\frac{3}{4}$.	(8) $\frac{1}{3}\times 4$.	*Ans.* $1\frac{1}{3}$.
(2) $\frac{1}{3}$ of 4.	„ $1\frac{1}{3}$.	(9) $\frac{1}{4}\times 3$.	„ $\frac{3}{4}$.
(3) $\frac{1}{3}$ of $\frac{1}{4}$.	„ $\frac{1}{12}$.	(10) $\frac{1}{4}\times 2$.	„ $\frac{1}{2}$.
(4) $\frac{1}{3}$ of $\frac{3}{4}$.	„ $\frac{1}{4}$.	(11) $\frac{3}{4}\times 2$.	„ $1\frac{1}{2}$.
(5) $\frac{1}{4}$ of $\frac{1}{3}$.	„ $\frac{1}{12}$.	(12) $\frac{3}{4}\times 3$.	„ $2\frac{1}{4}$.
(6) $\frac{1}{4}$ of $1\frac{1}{3}$.	„ $\frac{1}{3}$.	(13) $3\div 4$.	„ $\frac{3}{4}$.
(7) $\frac{1}{4}$ of $1\frac{1}{5}$.	„ $\frac{3}{10}$.	(14) $4\div 3$.	„ $1\frac{1}{3}$.

(15) $\frac{1}{3} \div 4$. *Ans.* $\frac{1}{12}$.
(16) $\frac{1}{4} \div 3$. " $\frac{1}{12}$.
(17) $\frac{3}{4} \div 3$. " $\frac{1}{4}$.
(18) $1\frac{1}{3} \div 4$. " $\frac{1}{3}$.
(19) $1\frac{2}{3} \div 4$. " $\frac{5}{12}$.
(20) $\frac{1}{7}$ of 6. " $\frac{6}{7}$.
(21) $\frac{1}{6}$ of 7. " $1\frac{1}{6}$.
(22) $\frac{1}{6}$ of $\frac{1}{7}$. " $\frac{1}{42}$.
(23) $\frac{1}{7}$ of $\frac{1}{6}$. " $\frac{1}{42}$.
(24) $\frac{1}{7}$ of $\frac{5}{6}$. " $\frac{5}{42}$.
(25) $\frac{1}{7}$ of $5\frac{3}{5}$. " $\frac{4}{5}$.
(26) $\frac{1}{7}$ of $5\frac{2}{5}$. " $\frac{27}{35}$.
(27) $5\frac{3}{5} \div 7$. " $\frac{4}{5}$.
(28) $5\frac{2}{5} \div 7$. " $\frac{27}{35}$.
(29) $5\frac{3}{5} \div \frac{2}{5}$. " 14.
(30) $5\frac{3}{5} \div 1\frac{2}{5}$. " 4.
(31) $6 \div 7$. " $\frac{6}{7}$.
(32) $7 \div 6$. " $1\frac{1}{6}$.
(33) $\frac{1}{7} \div 6$. " $\frac{1}{42}$.
(34) $\frac{1}{6} \div 7$. " $\frac{1}{42}$.
(35) $\frac{1}{7} \times 6$. " $\frac{6}{7}$.
(36) $\frac{1}{6} \times 7$. " $1\frac{1}{6}$.
(37) $\frac{1}{11}$ of 13. " $1\frac{2}{11}$.
(38) $\frac{1}{13}$ of $\frac{1}{11}$. " $\frac{1}{143}$.
(39) $\frac{1}{13}$ of 11. " $\frac{11}{13}$.
(40) $\frac{1}{11}$ of $\frac{1}{13}$. " $\frac{1}{143}$.
(41) $\frac{1}{13}$ of $\frac{4}{11}$. " $\frac{4}{143}$.
(42) $\frac{7}{13}$ of $\frac{4}{11}$. " $\frac{28}{143}$.
(43) $11 \div 13$. " $\frac{11}{13}$.
(44) $13 \div 11$. " $1\frac{2}{11}$.
(45) $\frac{1}{13} \div 11$. " $\frac{1}{143}$.
(46) $\frac{7}{13} \div 11$. " $\frac{7}{143}$.
(47) $\frac{11}{13} \div 11$. " $\frac{1}{13}$.
(48) $\frac{11}{13} \div \frac{1}{13}$. " 11.
(49) $\frac{1}{5}$ of 8. " $1\frac{3}{5}$.
(50) $\frac{1}{8}$ of 5. " $\frac{5}{8}$.

(51) $\frac{1}{8}$ of $\frac{1}{5}$. *Ans.* $\frac{1}{40}$.
(52) $\frac{1}{5}$ of $\frac{1}{8}$. " $\frac{1}{40}$.
(53) $\frac{1}{5}$ of $\frac{5}{8}$. " $\frac{1}{8}$.
(54) $\frac{1}{5}$ of $\frac{3}{8}$. " $\frac{3}{40}$.
(55) $\frac{3}{5}$ of $\frac{3}{8}$. " $\frac{9}{40}$.
(56) $\frac{4}{5}$ of $\frac{5}{8}$. " $\frac{1}{2}$.
(57) $\frac{1}{2}$ of $\frac{3}{4}$. " $\frac{3}{8}$.
(58) $\frac{5}{6}$ of $\frac{3}{7}$. " $\frac{5}{14}$.
(59) $\frac{2}{3}$ of $\frac{4}{5}$. " $\frac{8}{15}$.
(60) $\frac{1}{12}$ of 10. " $\frac{5}{6}$.
(61) $\frac{1}{10}$ of 12. " $1\frac{1}{5}$.
(62) $\frac{1}{12}$ of $\frac{1}{10}$. " $\frac{1}{120}$.
(63) $\frac{1}{10}$ of $\frac{1}{12}$. " $\frac{1}{120}$.
(64) $\frac{1}{12}$ of $\frac{7}{10}$. " $\frac{7}{120}$.
(65) $\frac{1}{10}$ of $\frac{7}{12}$. " $\frac{7}{120}$.
(66) $\frac{1}{10}$ of $2\frac{11}{12}$. " $\frac{7}{24}$.
(67) $\frac{1}{12}$ of $4\frac{4}{5}$. " $\frac{2}{5}$.
(68) $\frac{5}{12}$ of $4\frac{4}{5}$. " 2.
(69) $\frac{7}{12}$ of $4\frac{4}{5}$. " $2\frac{4}{5}$.
(70) $\frac{1}{12} \div 10$. " $\frac{1}{120}$.
(71) $\frac{1}{10} \div 12$. " $\frac{1}{120}$.
(72) $10 \div 12$. " $\frac{5}{6}$.
(73) $12 \div 10$. " $1\frac{1}{5}$.
(74) $4\frac{4}{5} \div 12$. " $\frac{2}{5}$.
(75) $4\frac{4}{5} \div 8$. " $\frac{3}{5}$.
(76) $4\frac{4}{5} \div \frac{3}{5}$. " 8.
(77) $4\frac{4}{5} \div 7$. " $\frac{24}{35}$.
(78) $4\frac{4}{5} \div 1\frac{3}{5}$. " 3.
(79) $4\frac{4}{5} \div \frac{2}{5}$. " 12.
(80) $4\frac{4}{5} \div 2$. " $2\frac{2}{5}$.
(81) $6 \div 9$. " $\frac{2}{3}$.
(82) $9 \div 6$. " $1\frac{1}{2}$.
(83) $\frac{1}{9}$ of 6. " $\frac{2}{3}$.
(84) $\frac{1}{6}$ of 9. " $1\frac{1}{2}$.
(85) $\frac{1}{6}$ of $\frac{1}{9}$. " $\frac{1}{54}$.
(86) $\frac{1}{6}$ of $\frac{5}{9}$. " $\frac{5}{54}$.

(87) $\frac{1}{6}$ of $\frac{5}{9}$ of 27.	*Ans.* $2\frac{1}{2}$.	(101) $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{11}$.	*Ans.* $\frac{56}{165}$.
(88) $\frac{1}{6}$ of $\frac{5}{9}$ of 45.	„ $4\frac{1}{6}$.	(102) $\frac{4}{5}$ of $\frac{7}{11}$ of $\frac{2}{3}$.	„ $\frac{56}{165}$.
(89) $\frac{1}{4}$ of $\frac{5}{9}$.	„ $\frac{5}{36}$.	(103) $\frac{4}{5}$ of $\frac{2}{3}$ of $\frac{7}{11}$.	„ $\frac{56}{165}$.
(90) $\frac{5}{9}$ of $\frac{1}{4}$.	„ $\frac{5}{36}$.	(104) $\frac{7}{11}$ of $\frac{2}{3}$ of $\frac{4}{5}$.	„ $\frac{56}{165}$.
(91) $\frac{5}{9}$ of 4.	„ $2\frac{2}{9}$.	(105) $\frac{1}{15}$ of $\frac{1}{12}$.	„ $\frac{1}{180}$.
(92) $\frac{1}{6}$ of $\frac{5}{9}$ of 4.	„ $\frac{10}{27}$.	(106) $\frac{1}{12}$ of $\frac{1}{15}$.	„ $\frac{1}{180}$.
(93) $\frac{5}{8}$ of 20.	„ $12\frac{1}{2}$.	(107) $\frac{1}{15}$ of 12.	„ $\frac{4}{5}$.
(94) $\frac{1}{7}$ of $\frac{5}{8}$ of 20.	„ $1\frac{11}{14}$.	(108) $\frac{1}{12}$ of 15.	„ $1\frac{1}{4}$.
(95) $\frac{3}{7}$ of $\frac{5}{8}$ of 20.	„ $5\frac{5}{14}$.	(109) $\frac{1}{23}$ of 46.	„ 2.
(96) $\frac{4}{5}$ of $\frac{7}{11}$.	„ $\frac{28}{55}$.	(110) $\frac{1}{46}$ of 23.	„ $\frac{1}{2}$.
(97) $\frac{1}{2}$ of $\frac{4}{5}$ of $\frac{7}{11}$.	„ $\frac{14}{55}$.	(111) $\frac{1}{60}$ of 320.	„ $5\frac{1}{3}$.
(98) $\frac{3}{5}$ of $\frac{7}{11}$.	„ $\frac{21}{55}$.	(112) $\frac{1}{320}$ of 60.	„ $\frac{3}{16}$.
(99) $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{7}{11}$.	„ $\frac{21}{110}$.	(113) $\frac{4}{5}$ of $\frac{3}{11}$.	„ $\frac{12}{55}$.
(100) $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{7}{11}$.	„ $\frac{14}{55}$.	(114) $\frac{3}{8}$ of $\frac{7}{10}$.	„ $\frac{21}{80}$.

EXERCISE XVIII.

Prove by finding values that:

(1) $\frac{5}{7}$ of £$\frac{3}{8}$ = $\frac{3}{8}$ of £$\frac{5}{7}$.

(2) $\frac{5}{7}$ of $\frac{3}{8}$ of 1 cwt. = $\frac{3}{8}$ of $\frac{5}{7}$ of 1 cwt.

(3) $\frac{5}{7}$ of $\frac{3}{8}$ of 1 yard = $\frac{3}{8}$ of $\frac{5}{7}$ of 1 yard.

(4) $\frac{2}{3}$ of $\frac{7}{11}$ of 1 mile = $\frac{7}{11}$ of $\frac{2}{3}$ of 1 mile.

§ 6. Simplify $\frac{4}{9}$ of $\frac{8}{11}$ of $\frac{12}{13}$ of 20.

$$\tfrac{4}{9} \text{ of } \tfrac{8}{11} = \frac{4\times 8}{9\times 11}$$

$$\tfrac{4}{9} \text{ of } \tfrac{8}{11} \text{ of } \tfrac{12}{13} = \frac{4\times 8\times 12}{9\times 11\times 13}$$

$$\tfrac{4}{9} \text{ of } \tfrac{8}{11} \text{ of } \tfrac{12}{13} \text{ of } 20 = \frac{4\times 8\times 12}{9\times 11\times 13} \text{ of } 20.$$

Since $\frac{1}{4}$ of 3 = $\frac{3}{4}$ of 1 (Part II. Ch. II. § 2), $\frac{1}{9\times 11\times 13}$ of $20 = \frac{20}{9\times 11\times 13}$, and $\frac{4\times 8\times 12}{9\times 11\times 13}$ of $20 = \frac{4\times 8\times 12\times 20}{9\times 11\times 13} = \frac{7680}{1287} = 5\frac{1245}{1287}$.

Learn by heart: *To simplify a complex fraction, multiply together all the numerators for a new numerator, and all the denominators for a new denominator.*

Simplify $\frac{4}{21}$ of $\frac{9}{20}$ of $\frac{10}{11}$ of $\frac{15}{16}$ of $\frac{14}{25}$ of 11.

$$\tfrac{4}{21} \text{ of } \tfrac{9}{20} \text{ of } \tfrac{10}{11} \text{ of } \tfrac{15}{16} \text{ of } \tfrac{14}{25} \text{ of } 11 = \frac{4 \times 9 \times 10 \times 15 \times 14 \times 11}{21 \times 20 \times 11 \times 16 \times 25}$$

$$400)\frac{831600}{1848000} \Big| \; 3)\frac{2079}{4620} \Big| \; 11)\frac{693}{1540} \Big| \; 7)\frac{63}{140} \Big| \; \frac{9}{20}$$

This reduction to lowest terms might have been performed before actually multiplying the several numerators and denominators.

Since 4×9 is 3 times as much as 4×3, $4 \times 9 \times 10 \times 15 \times 14 \times 11$ is 3 times as much as $4 \times 3 \times 15 \times 14 \times 11$. Hence, dividing by 3 the one factor 9, divides the whole numerator by 3; and generally:—*A product is divided by any number if* ONE *of its factors is divided by that number*, and therefore any one factor of the numerator may be divided by a number, provided some one factor of the denominator is also divided by that number.

Note that this rule applies only to a series of factors, and not to a series of *addenda*, where *each* number must be divided in order to divide the *sum*.

The several numerators and denominators may therefore be "cancelled" against each other, thus:

4 and 16	become respectively	1 and 4
9 and 21	"	3 and 7
15 and 25	"	3 and 5
10 and 20	"	1 and 2
11 and 11	"	1 and 1

and the fraction will be

$$\frac{\overset{1}{\cancel{4}}}{\underset{7}{\cancel{21}}} \text{ of } \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{20}}} \text{ of } \frac{\overset{1}{\cancel{10}}}{\underset{1}{\cancel{11}}} \text{ of } \frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{16}}} \text{ of } \frac{14}{\underset{5}{\cancel{25}}} \text{ of } \overset{1}{\cancel{11}} = \frac{1 \times 3 \times 1 \times 3 \times 14 \times 1}{7 \times 2 \times 1 \times 4 \times 5}$$

which admits of further cancelling. 7 and 14 become respectively 1 and 2; and this 2 cancels against the 2 in the denominator, leaving $\frac{1 \times 3 \times 1 \times 3 \times 1 \times 1}{1 \times 1 \times 1 \times 4 \times 5} = \frac{9}{20}$ as before.

Mod. op.:

$$\dfrac{\overset{1}{\cancel{4}}}{\underset{\underset{1}{\cancel{7}}}{\cancel{21}}} \text{ of } \dfrac{\overset{3}{\cancel{9}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{28}}} \text{ of } \dfrac{\overset{1}{\cancel{10}}}{\underset{1}{\cancel{11}}} \text{ of } \dfrac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{16}}} \text{ of } \dfrac{\overset{\overset{1}{\cancel{2}}}{\cancel{14}}}{\underset{5}{\cancel{25}}} \text{ of } \overset{1}{\cancel{11}} = \dfrac{9}{20}$$

Simplify $\frac{5}{8}$ of $\frac{3}{7}$ of $\frac{16}{25}$ of $\frac{4}{9}$ of $\frac{21}{55}$ of $6\frac{5}{12}$.

$$\dfrac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{8}}} \text{ of } \dfrac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{7}}} \text{ of } \dfrac{\overset{2}{\cancel{16}}}{\underset{5}{\cancel{25}}} \text{ of } \dfrac{\overset{1}{\cancel{4}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{9}}} \text{ of } \dfrac{\overset{\overset{1}{\cancel{3}}}{\cancel{21}}}{\underset{5}{\cancel{55}}} \text{ of } \dfrac{\overset{7}{\cancel{77}}}{\underset{3}{\cancel{12}}} = \dfrac{14}{75}$$

EXERCISE XIX.

(1) $\frac{2}{3}$ of $\frac{7}{11}$.

(2) $\frac{2}{3}$ of $\frac{5}{6}$.

(3) $\frac{3}{7}$ of $2\frac{1}{10}$.

(4) $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of 4.

(5) $\frac{3}{5}$ of $\frac{10}{27}$ of $\frac{9}{20}$ of $8\frac{1}{3}$.

(6) $\frac{8}{11}$ of $\frac{20}{21}$ of $\frac{35}{48}$ of $2\frac{10}{19}$.

(7) $\frac{8}{9}$ of $\frac{36}{37}$ of $\frac{5}{18}$ of 111.

(8) $\frac{7}{12}$ of $\frac{28}{29}$ of $\frac{26}{35}$ of $7\frac{1}{4}$.

(9) $\frac{2}{3}$ of $\frac{2}{3}$ of $\frac{2}{3}$ of $\frac{2}{3}$.

(10) $\frac{7}{25}$ of $3\frac{4}{7}$.

(11) $\frac{42}{43}$ of $\frac{13}{105}$ of $1\frac{7}{208}$.

(12) $\frac{5}{8}$ of $\frac{120}{121}$ of $\frac{66}{85}$ of 17.

(13) $\frac{14}{15}$ of $\frac{23}{35}$.

(14) $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{1}{4}$ of $\frac{1}{5}$ of $\frac{1}{6}$.

(15) $\frac{38}{39}$ of $\frac{52}{57}$ of $\frac{69}{86}$ of $1\frac{20}{23}$.

(16) $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$ of $\frac{9}{10}$ of 10.

(17) $\frac{1}{10}$ of $\frac{1}{10}$ of $\frac{1}{10}$ of $\frac{1}{10}$ of $\frac{1}{10}$.

(18) $\frac{1}{10}$ of $\frac{2}{9}$ of $\frac{3}{8}$ of $\frac{4}{7}$ of $\frac{5}{6}$ of $1\frac{1}{5}$.

(19) $\frac{5}{11}$ of $\frac{34}{35}$ of $1\frac{4}{51}$.

(20) $\frac{7}{25}$ of $\frac{8}{11}$ of 30.

(21) $\frac{3}{5}$ of $\frac{6}{7}$ of $\frac{3}{11}$ of 4.

(22) $\frac{2}{3}$ of $\frac{4}{9}$ of $\frac{8}{27}$.

(23) $\frac{2}{21}$ of $\frac{7}{9}$ of $\frac{5}{8}$ of $\frac{4}{15}$ of 12.

(24) $\frac{8}{15}$ of $\frac{2}{3}$ of $\frac{12}{13}$ of $\frac{5}{9}$ of 39.

(25) $\frac{8}{15}$ of $\frac{9}{16}$ of $\frac{10}{33}$ of $\frac{11}{21}$ of $\frac{12}{35}$ of $\frac{13}{15}$ of $\frac{14}{39}$ of 20.

(26) $\frac{1}{7}$ of $20 + \frac{3}{7}$ of 61.

(27) $\frac{2}{11}$ of 7 — $\frac{1}{22}$ of 6.

(28) $\frac{3}{13}$ of $2 + \frac{3}{13}$ of $3 + \frac{3}{13}$ of $4 + \frac{3}{13}$ of $5 + \frac{3}{13}$ of 6.

(29) $\frac{2}{5}$ of $2\frac{1}{2} + \frac{3}{7}$ of $2\frac{1}{3} + \frac{5}{9}$ of $1\frac{4}{5} + \frac{7}{26}$ of $3\frac{5}{7}$.

(30) $\frac{5}{12}$ of $16 + \frac{11}{12}$ of $20 + \frac{7}{12}$ of $36 + \frac{1}{12}$ of 8.

(31) $\frac{113}{355}$ of $\frac{85}{226}$ of $\frac{12}{35}$ of $1\frac{3}{4}$.

(32) $\frac{7}{8}$ of $\frac{3}{4}$ of $\frac{8}{21}$ of $\frac{4}{5}$ of $\frac{5}{8}$ of $\frac{3}{4}$ of 8.

(33) $\frac{2}{13}$ of $\frac{39}{40}$ of $\frac{52}{117}$.

(34) $\frac{9}{11}$ of $\frac{7}{12}$ of $\frac{22}{63}$ of 48.

(35) $\frac{28}{59}$ of $\frac{3}{8}$ of $\frac{59}{63}$.

(36) $\frac{39}{40}$ of $\frac{7}{65}$ of $\frac{25}{42}$ of 12.

CHAPTER III.

DIFFERENT DENOMINATORS.

§ 1. Add $\frac{1}{2}$ and $\frac{1}{3}$. "In Addition and Subtraction we must always have the *same kind* of units" (Part I. Ch. V. § 17); hence $\frac{1}{2}$ and $\frac{1}{3}$ cannot be added in their present shape.

Add $\frac{1}{2}$ and $\frac{1}{4}$. *Ans.* $\frac{3}{4}$, because $\frac{1}{2} = \frac{2}{4}$. Can we not similarly reduce $\frac{1}{2}$ into thirds? A whole has 3 thirds, therefore $\frac{1}{2} = \frac{1}{3} + \frac{1}{2}$ of $\frac{1}{3} = \frac{1}{3} + \frac{1}{6}$, therefore $\frac{1}{2} + \frac{1}{3} = (\frac{1}{3} + \frac{1}{6}) + \frac{1}{3} = \frac{2}{3} + \frac{1}{6}$. But as a whole is 6 sixths, $\frac{1}{3} = \frac{2}{6}$ and $\frac{2}{3} = \frac{4}{6}$; $\therefore \frac{1}{2} + \frac{1}{3} = \frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$.* The result, however, can be obtained by a simpler process.

$$1 = \tfrac{2}{2} = \tfrac{3}{3} = \tfrac{4}{4} = \tfrac{5}{5} = \tfrac{6}{6} = \tfrac{7}{7} = \tfrac{8}{8} = \tfrac{9}{9} = \tfrac{10}{10} = \tfrac{11}{11} = \tfrac{12}{12} = \tfrac{13}{13} = \tfrac{14}{14} = \tfrac{15}{15} = \tfrac{16}{16} = \tfrac{17}{17} = \tfrac{18}{18},$$

$$\therefore \tfrac{1}{2} = \tfrac{2}{4} = \tfrac{3}{6} = \tfrac{4}{8} = \tfrac{5}{10} = \tfrac{6}{12} = \tfrac{7}{14} = \tfrac{8}{16} = \tfrac{9}{18},$$

$$\tfrac{1}{3} = \tfrac{2}{6} = \tfrac{3}{9} = \tfrac{4}{12} = \tfrac{5}{15} = \tfrac{6}{18}.$$

We see from this that halves and thirds can both be converted into sixths, twelfths, eighteenths, &c.,

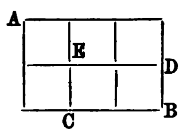
$$\therefore \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{3}{6} + \tfrac{2}{6} = \tfrac{5}{6},$$

$$\text{or} \quad = \tfrac{6}{12} + \tfrac{4}{12} = \tfrac{10}{12} = \tfrac{5}{6},$$

$$\text{or} \quad = \tfrac{9}{18} + \tfrac{6}{18} = \tfrac{15}{18} = \tfrac{5}{6},$$

and so on with other common multiples of 2 and 3. But the L.C.M. is obviously the best. Illustrations:

a)



$AB = 1$
$AC = \frac{1}{3}$
$AD = \frac{1}{2}$
$AE = \frac{1}{6}$

$$\therefore AC = \tfrac{2}{6}, \quad AD = \tfrac{3}{6}; \ \therefore \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6}.$$

b) $\frac{1}{2}$ of 1*s.* = 6*d.*, $\frac{1}{3}$ of 1*s.* = 4*d.* $(\frac{1}{2} + \frac{1}{3})$ of 1*s.* = 6*d.* + 4*d.* = 10*d.* = $\frac{5}{6}$ of 1*s.*

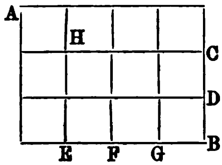
* This mode of reasoning is here given historically, as being that most commonly suggested by the more intelligent pupils.

Exercise L.

To be worked mentally from diagrams drawn on the black board.

(1) $\frac{1}{2}+\frac{1}{3}$. *Ans.* $\frac{5}{6}$.
(2) $\frac{1}{2}-\frac{1}{3}$. „ $\frac{1}{6}$.
(3) $\frac{1}{2}\div\frac{1}{6}$. „ 3 times.
(4) $\frac{1}{3}\div\frac{1}{6}$. „ Twice.
(5) $\frac{1}{6}\times 3$. „ $\frac{3}{6}$ or $\frac{1}{2}$.
(6) $\frac{1}{6}\times 2$. *Ans.* $\frac{2}{6}$ or $\frac{1}{3}$.
(7) $\frac{2}{3}\div\frac{1}{6}$. „ 4 times.
(8) $(\frac{2}{3}+\frac{1}{2})\div\frac{1}{6}$. 7 times.
(9) $\frac{2}{3}+\frac{1}{2}$. „ $\frac{7}{6}=1\frac{1}{6}$.
(10) $\frac{2}{3}-\frac{1}{2}$. „ $\frac{1}{6}$.

(11) A post is driven through water into the mud below; $\frac{1}{2}$ is buried in the mud, $\frac{1}{3}$ is under water, and there are 5 feet above water. Find the length of the post. *Ans.* 30 feet.



(12) $\frac{1}{3}+\frac{1}{4}$. *Ans.* $\frac{7}{12}$.
(13) $\frac{1}{3}-\frac{1}{4}$. „ $\frac{1}{12}$.
(14) $\frac{3}{4}+\frac{2}{3}$. „ $1\frac{5}{12}$.
(15) $\frac{3}{4}-\frac{2}{3}$. „ $\frac{1}{12}$.
(16) $\frac{1}{3}$ of $\frac{1}{4}$. „ $\frac{1}{12}$.
(17) $\frac{1}{4}$ of $\frac{1}{3}$. *Ans.* $\frac{1}{12}$.
(18) $\frac{2}{3}\div\frac{1}{12}$. „ 8 times.
(19) $\frac{3}{4}\div\frac{1}{12}$. „ 9 times.
(20) $1\frac{3}{4}+1\frac{1}{3}$. „ $3\frac{1}{12}$.
(21) $1\frac{3}{4}-1\frac{2}{3}$. „ $\frac{1}{12}$.

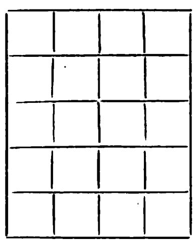
Reduce to lowest terms:

(22) $\frac{2}{12}$. *Ans.* $\frac{1}{6}$. (24) $\frac{4}{12}$. *Ans.* $\frac{1}{3}$. (26) $\frac{8}{12}$. *Ans.* $\frac{2}{3}$. (28) $\frac{10}{12}$. *Ans.* $\frac{5}{6}$.
(23) $\frac{3}{12}$. „ $\frac{1}{4}$. (25) $\frac{6}{12}$. „ $\frac{1}{2}$. (27) $\frac{9}{12}$. „ $\frac{3}{4}$.

(29) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$. *Ans.* $1\frac{3}{12}|\frac{1}{4}$.
(30) $1\frac{1}{2}+2\frac{2}{3}+3\frac{1}{4}+2\frac{5}{6}$. *Ans.* $10\frac{3}{12}|\frac{1}{4}$.

(31) If $(\frac{1}{4}+\frac{1}{3})$ of a sum of money is £3. 10s., what is the whole sum? *Ans.* £6.

(32) On Monday I spent $\frac{2}{3}$ of my money, on Tuesday $\frac{1}{4}$ of it, and had then £1. 10s. left. How much had I at first? *Ans.* £18.



(33) $\frac{1}{4}+\frac{1}{5}$. *Ans.* $\frac{9}{20}$.
(34) $\frac{1}{4}-\frac{1}{5}$. „ $\frac{1}{20}$.
(35) $\frac{1}{4}$ of $\frac{1}{5}$. „ $\frac{1}{20}$.
(36) $\frac{1}{5}$ of $\frac{1}{4}$. „ $\frac{1}{20}$.
(37) $\frac{3}{4}+\frac{2}{5}$. „ $1\frac{3}{20}$.
(38) $\frac{3}{4}-\frac{2}{5}$. *Ans.* $\frac{7}{20}$.
(39) $\frac{3}{4}$ of $\frac{2}{5}$. „ $\frac{3}{10}$.
(40) $\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{10}+\frac{1}{20}$. „ $1\frac{1}{10}$.
(41) $2\frac{1}{2}+1\frac{3}{4}+3\frac{2}{5}$. „ $7\frac{13}{20}$.
(42) Express $\frac{2}{5}$ as a fraction whose denominator is 20. *Ans.* $\frac{8}{20}$.
(43) „ $\frac{3}{10}$ „ „ „ $\frac{6}{20}$.
(44) „ $\frac{1}{2}$ „ „ „ $\frac{10}{20}$.
(45) „ $\frac{3}{4}$ „ „ „ $\frac{15}{20}$.
(46) „ $\frac{7}{10}$ „ „ „ $\frac{14}{20}$.
(47) $2\frac{1}{5}\div\frac{1}{20}$. *Ans.* 44 times.
(48) $2\frac{1}{4}-1\frac{3}{5}$. *Ans.* $\frac{13}{20}$.
(49) $13\frac{1}{2}-(1\frac{3}{4}+2\frac{2}{5}+1\frac{7}{10})$. „ $7\frac{13}{20}$.
(50) $(\frac{1}{4}+\frac{1}{5})+(\frac{1}{4}-\frac{1}{5})+(\frac{1}{4}$ of $\frac{1}{5})$. „ $\frac{11}{20}$.

Reduce to lowest terms:

(51) $\frac{18}{20}$. *Ans.* $\frac{9}{10}$. (53) $\frac{15}{20}$. *Ans.* $\frac{3}{4}$. (55) $\frac{10}{20}$. *Ans.* $\frac{1}{2}$. (57) $\frac{4}{20}$. *Ans.* $\frac{1}{5}$.
(52) $\frac{16}{20}$. „ $\frac{4}{5}$. (54) $\frac{12}{20}$. „ $\frac{3}{5}$. (56) $\frac{5}{20}$. „ $\frac{1}{4}$. (58) $\frac{8}{20}$. „ $\frac{2}{5}$.

Exercise XX.

Shew by finding values that:

(1) $(\frac{1}{2}+\frac{1}{3})$ of £1 $=\frac{5}{6}$ of £1.
(2) „ 13*s.* 9*d.* $=\frac{5}{6}$ of 13*s.* 9*d.*
(3) „ 1 oz. troy $=\frac{5}{6}$ of 1 oz. troy.
(4) „ 1 yard $=\frac{5}{6}$ of 1 yard.
(5) $(\frac{3}{4}+\frac{2}{3})$ of £1 $=1\frac{5}{12}$ of £1.
(6) „ 13*s.* 9*d.* $=1\frac{5}{12}$ of 13*s.* 9*d.*
(7) „ 1 oz. troy $=1\frac{5}{12}$ of 1 oz. troy.
(8) „ 1 yard $=1\frac{5}{12}$ of 1 yard.
(9) „ £2. 11*s.* 6*d.* $=1\frac{5}{12}$ of £2. 11*s.* 6*d.*
(10) $(\frac{3}{4}-\frac{2}{5})$ of £1 $=\frac{7}{20}$ of £1.
(11) $(\frac{4}{5}+\frac{1}{4})$ of 1 oz. troy $=1\frac{1}{20}$ oz. troy.

§ 2. $\frac{5}{8}+\frac{7}{12}$. L.C.M. of 8 and 12 is 24.

$$1=\frac{24}{24};\quad \frac{1}{8}=\frac{3}{24};\quad \frac{5}{8}=\frac{15}{24}$$
$$\frac{1}{12}=\frac{2}{24};\quad \frac{7}{12}=\frac{14}{24}$$
$$\therefore\ \frac{5}{8}+\frac{7}{12}=\frac{15+14}{24}=\frac{29}{24}=1\frac{5}{24}$$

Mod. op.:

$$\begin{array}{r|l} \multicolumn{2}{c}{\underbrace{24}} \\ 3 & 15 \\ 2 & 14 \\ \hline \multicolumn{2}{c}{\frac{29}{24}=1\frac{5}{24}} \end{array}$$

Wording: 8 in 24, 3′; 3×5=15′; 12 in 24, 2′; 7×2=14′, &c.

$\frac{1}{2}+\frac{2}{3}+\frac{4}{9}+\frac{7}{12}+\frac{5}{6}$. L.C.M. of 2, 3, 9, 12, 6, is 36.

$$\begin{array}{r|l} \multicolumn{2}{c}{\underbrace{36}} \\ — & 18 \\ 12 & 24 \\ 4 & 16 \\ 3 & 21 \\ 6 & 30 \\ \hline \multicolumn{2}{c}{36)\,109\,(3\frac{1}{36}} \\ \multicolumn{2}{c}{1} \end{array}$$

$8\frac{13}{48}+19\frac{31}{45}+429\frac{17}{18}+\frac{17}{60}$. L.C.M. of 48, 45, 18, 60 = 720.

$$\begin{array}{r r|l} & \multicolumn{2}{c}{\underbrace{720}} \\ 8 & 15 & 195 \\ 19 & 16 & 496 \\ 429 & 40 & 680 \\ & 12 & 204 \\ \hline 456 & \multicolumn{2}{c}{\frac{1575}{720}\Big|\frac{175}{80}\Big|\frac{35}{16}=2\frac{3}{16}.} \end{array}$$

Ans. $458\frac{3}{16}$.

$$\frac{1}{2}\text{ of }\frac{4}{5}+\frac{2}{3}\text{ of }\frac{7}{8}+\frac{1}{12}\text{ of }8+\frac{3}{8}\div 5+\frac{7}{15}\times 4+\frac{49}{16}$$
$$=\frac{2}{5}+\frac{7}{12}+\frac{2}{3}+\frac{3}{40}+1\frac{13}{15}+3\frac{1}{16}.$$

L.C.M. of 5, 12, 3, 40, 15, 16 = 240.

$$\begin{array}{r r|l} & \multicolumn{2}{c}{\underbrace{240}} \\ & 48 & 96 \\ & 20 & 140 \\ & 80 & 160 \\ & 6 & 18 \\ 1 & 16 & 208 \\ 3 & — & 15 \\ \hline 4 & \multicolumn{2}{c}{240)\,637\,(2} \\ & \multicolumn{2}{c}{157} \end{array}$$

Ans. $6\frac{157}{240}$.

$$9\frac{5}{6}-7\frac{3}{8}=9\frac{20}{24}-7\frac{9}{24}=2\frac{11}{24}$$
$$7\frac{5}{21}-2\frac{11}{14}=7\frac{10}{42}-2\frac{33}{42}=4\frac{19}{42}$$

EXERCISE XXI.

(1) $\frac{4}{5}+\frac{5}{6}$; $\frac{3}{4}+\frac{7}{8}$; $\frac{1}{2}+\frac{1}{6}$; $\frac{1}{2}-\frac{1}{6}$; $\frac{6}{7}-\frac{4}{9}$; $\frac{4}{15}+\frac{11}{20}$; $\frac{5}{16}+\frac{11}{24}$.

(2) $12\frac{5}{8}+7\frac{3}{16}$; $12\frac{5}{8}-7\frac{3}{16}$; $85\frac{7}{12}+27\frac{11}{18}$; $85\frac{7}{12}-27\frac{11}{18}$.

(3) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$; $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$; $\frac{5}{6}+\frac{11}{12}+\frac{8}{15}+\frac{7}{20}+\frac{13}{30}$.

(4) $5\frac{17}{20}+11\frac{19}{30}+24\frac{21}{40}+\frac{9}{50}+17\frac{8}{15}+14+11\frac{5}{12}$.

(5) $9\frac{4}{7}+15\frac{11}{28}+103\frac{17}{63}+1\frac{11}{42}+10\frac{1}{4}$.

(6) $1473-279$; $1473\frac{5}{18}-279$; $1473-279\frac{11}{12}$; $1473\frac{5}{18}-279\frac{11}{12}$; $1473\frac{7}{18}-279\frac{11}{12}$.

(7) $\frac{5}{14}+7\frac{8}{25}+11\frac{9}{16}+10\frac{11}{30}+14\frac{5}{8}+100+77\frac{6}{35}$.

(8) $\frac{5}{14}+\frac{6}{11}+9\frac{1}{2}$; $20\frac{5}{12}+11\frac{7}{20}+5\frac{1}{8}+305$; $278\frac{15}{16}-30\frac{5}{12}$.

(9) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}$; $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\frac{7}{8}+\frac{8}{9}+\frac{9}{10}$.

(10) $\frac{1}{2}-\frac{1}{3}$; $\frac{1}{3}-\frac{1}{4}$; $\frac{1}{4}-\frac{1}{5}$; $\frac{1}{5}-\frac{1}{6}$; $\frac{1}{6}-\frac{1}{7}$; $\frac{1}{7}-\frac{1}{8}$; $\frac{1}{8}-\frac{1}{9}$; $\frac{1}{9}-\frac{1}{10}$.

(11) $\frac{5}{17}+\frac{11}{34}+\frac{14}{51}+\frac{19}{68}$; $\frac{11}{38}+\frac{14}{57}+\frac{17}{76}$; $\frac{9}{13}+\frac{5}{39}+\frac{17}{52}$.

(12) $118\frac{5}{11}-17\frac{3}{14}$; $94\frac{5}{11}-91\frac{13}{14}$; $125\frac{5}{22}-10\frac{17}{33}$; $40\frac{1}{2}-30\frac{47}{90}$.

(13) $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}$; $\frac{4}{7}+\frac{2}{9}-\frac{3}{5}+\frac{11}{21}$.

(14) $\frac{1}{2}$ of $\frac{1}{3}+\frac{1}{3}$ of $\frac{1}{4}$; $\frac{2}{5}$ of $\frac{10}{11}+\frac{3}{8}$ of $\frac{16}{33}$; $\frac{1}{12}$ of $11+\frac{1}{11}$ of 12.

(15) $\frac{2}{11}$ of $210+\frac{7}{12}$ of $210+\frac{5}{8}$ of $7\frac{1}{11}$.

(16) $3\frac{5}{24}+7\frac{11}{12}+8\frac{13}{16}+9\frac{14}{15}$.

(17) $\frac{1}{7}$ of $5\frac{4}{9}+\frac{2}{13}$ of $1\frac{2}{37}+\frac{2}{3}$ of $\frac{17}{37}$.

(18) $\frac{7}{20}$ of $63-\frac{3}{50}$ of $7\frac{1}{2}$.

(19) $\frac{15}{16}$ of $7\frac{3}{5}+\frac{3}{17}$ of $10\frac{1}{5}-\frac{5}{7}$ of $2\frac{9}{10}$.

(20) *a.* $\frac{3}{10}+\frac{7}{100}+\frac{9}{1000}+\frac{5}{10000}$; $\frac{8}{100}+\frac{17}{10000}+\frac{9}{100000}$.

b. $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\frac{1}{100000}$.

c. $\frac{1}{10}+\frac{2}{100}+\frac{3}{1000}+\frac{4}{10000}+\frac{5}{100000}$.

d. $\frac{1}{10}+\frac{1}{1000}+\frac{1}{100000}+\frac{1}{10000000}$.

e. $\frac{7}{10}+\frac{9}{100}+\frac{4}{10000}$.

f. $\frac{3}{100}+\frac{4}{1000}+\frac{7}{1000000}$.

g. $\frac{17}{10}+\frac{17}{100}+\frac{17}{1000}+\frac{17}{10000}+\frac{17}{100000}$.

h. $\frac{143}{100}+\frac{2471}{1000}+\frac{82643}{1000000}$.

(21) Which is the greater of each of the following pairs, and by how much: $\frac{5}{8}$ or $\frac{7}{11}$; $\frac{4}{17}$ or $\frac{15}{67}$; $\frac{4}{17}$ or $\frac{15}{68}$; $\frac{2}{11}$ or $\frac{9}{50}$?

(22) Also of $\frac{1}{2}$ of $\frac{1}{3}$ and $\frac{1}{4}$ of $\frac{3}{5}$; $\frac{2}{11}$ of $3\frac{1}{2}$ and $\frac{4}{11}$ of $1\frac{3}{4}$?

(23) Add together the sum and the difference of $\frac{9}{20}$ of $7\frac{3}{8}$, and $\frac{5}{14}$ of $3\frac{3}{40}$.

(24) Find the difference between $\frac{1}{3}$ and $\frac{3}{10}$; $\frac{4}{11}$ and $\frac{36}{100}$; $\frac{6}{7}$ and $\frac{857142}{1000000}$; $\frac{1}{960}$ and $\frac{1}{1000}$.

(25) $43\frac{7}{15} - 1\frac{1}{3} - 1\frac{31}{48} - 1\frac{23}{24} - 2\frac{13}{48} - 2\frac{7}{12} - 2\frac{43}{48} - 3\frac{5}{12}$.

(26) $43\frac{7}{15} - (1\frac{1}{3} + 1\frac{31}{48} + 1\frac{23}{24} + 2\frac{13}{48} + 2\frac{7}{12} + 2\frac{43}{48} + 3\frac{5}{12})$.

(27) $(\frac{1}{2} + \frac{4}{18} + 7\frac{9}{40} + 8\frac{14}{39} + 7\frac{1}{4} + 8\frac{3}{10} + 4\frac{1}{12}) - 36\frac{1}{40}$.

(28) $(8\frac{5}{18} + 9\frac{10}{27} + 17\frac{11}{36} + 40) - (30\frac{13}{40} + 11\frac{11}{20})$.

(29) $(172\frac{19}{78} + 93\frac{14}{117}) + (172\frac{19}{78} - 93\frac{14}{117})$.

(30) $(172\frac{19}{78} + 93\frac{14}{117}) - (172\frac{19}{78} - 93\frac{14}{117})$.

(31) $6\frac{3}{4} + \frac{27}{5} + \frac{5}{12} \times 3 + \frac{7}{18} \times 5 + 6\frac{2}{5} \div 4 + 1\frac{1}{8} \div 2 + \frac{5}{9}$ of $\frac{3}{4}$.

(32) A bequeathed to his two sons $\frac{1}{4}$ of his property each, to each of his three daughters $\frac{1}{9}$ of his property, to his nephew $\frac{1}{18}$ of his property, and a like sum to his niece; the remainder, £1000, to a hospital. Find the value of the whole property, and the shares in money of the several heirs.

(33) If for 6 days running a sailor spent each day half of what he had at the beginning of the day, and had then 6*s.* 6*d.* left, how much had he at first?

(34) In a cricket-match, 11 players made a certain number of runs; the first made $\frac{1}{10}$ of the total number, the next three each $\frac{5}{36}$, the next five each $\frac{1}{15}$, and the two last 18 runs between them. The other side made successively $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{15}$, $\frac{1}{20}$, $\frac{1}{30}$, $\frac{1}{40}$ and $\frac{1}{60}$ of their opponents' total. Which side had won, and by how much?

§ 3. Multiplication by Fractions.

£7. 13*s.* 6*d.* × $\frac{2}{3}$. We have shewn (Part I. Ch. V. § 17) that in multiplication one of the factors must be so many *times*. £7. 13*s.* 6*d.* cannot mean *times*; what sense, then, are we to attach to "$\frac{2}{3}$ times"? £7. 13*s.* 6*d.* × 5 may be interpreted: Find the cost of 5 articles at £7. 13*s.* 6*d.* each; and similarly, £7. 13*s.* 6*d.* × $\frac{2}{3}$ would be interpreted: Find the cost of $\frac{2}{3}$ of an article at £7. 13*s.* 6*d.* for one.

This cost will evidently be $\frac{5}{8}$ *of* £7. 13*s.* 6*d.* Are we, then, to conclude generally that × means *of?* Certainly not, until we have tested this extension of meaning by the two principles laid down above (Part I. Ch. V. § 15.)

(*a*) It *has* an intelligible meaning.

(*b*) It does "not alter the sense attached to the symbol in the earlier cases," for, calling £7. 13*s.* 6*d.* "a collection," multiplying that collection by 5 means taking 5 *of* them. Does it "remain subject to the general rules already established"? The only general rules hitherto established are that multiplication may be performed in any order; thus $5 \times 3 = 3 \times 5$; and that though one of the numbers must mean *times*, it is indifferent which does so; thus £5×3 = £3×5.

We have shewn above (Part II. Ch. II. § 5) that $\frac{5}{7}$ of $\frac{3}{8} = \frac{3}{8}$ of $\frac{5}{7}$, and it follows immediately that $\frac{5}{7}$ of £$\frac{3}{8} = \frac{3}{8}$ of £$\frac{5}{7}$, and we can also see that $\frac{2}{3}$ of £7. 13*s.* 6*d.* $= 7 \times$ £$\frac{2}{3} + 13 \times \frac{2}{3}s. + 6 \times \frac{2}{3}d.$

Hence, generally: $\times$ = OF.

N.B. Beginners are apt to connect the word *of* with division, but division can only be interpreted by "*of*" by altering the divisor; thus to divide by 2, 3, &c., is to take $\frac{1}{2}$, $\frac{1}{3}$, &c., and not 2, 3, &c., *of* the dividend.

We now see that multiplication of fractions by fractions is equivalent to simplification of complex fractions.

§ 4. We might have arrived at the same conclusion by a less elementary process of reasoning. To multipy by $\frac{2}{3}$ is to multiply by $\frac{1}{3}$ of 2; if, then, we multiply by 2, our product must be three times too great, and we must therefore divide it by 3. Hence, to multiply by $\frac{2}{3}$, we multiply by 2 and divide by 3, which is *taking* $\frac{2}{3}$ of it.

§ 5. We have hitherto connected multiplication with increase, agreeing therein with the derivation of the word; but we have already seen that this connection fails in the case of multiplication by 1, where the multiplicand remains unaltered. Compare:

£8 × 5 = 5 of £8 = £40.
£8 × 3 = 3 of £8 = £24.
£8 × 1 = 1 of £8 = £8.
£8 × $\frac{1}{2}$ = $\frac{1}{2}$ of £8 = £4.
£8 × $\frac{1}{3}$ = $\frac{1}{3}$ of £8 = £2. 13*s.* 4*d.*

We see that as the multiplier is diminished, the product is also diminished; when the multiplier becomes 1, the product = the multiplicand; and when the multiplier is less than 1, the product is *less* than the multiplicand.

$$\tfrac{2}{5}\times 1\tfrac{7}{8}\times 12\times 5\tfrac{1}{4}\times\tfrac{6}{7}\ (=\tfrac{2}{5}\text{ of }\tfrac{15}{8}\text{ of }12\text{ of }\tfrac{21}{4}\text{ of }\tfrac{6}{7})$$

$$=\frac{\overset{1}{\cancel{2}}\times\overset{3}{\cancel{15}}\times\overset{3}{\cancel{12}}\times\overset{3}{\cancel{21}}\times\overset{3}{\cancel{6}}}{\underset{1}{\cancel{5}}\times\underset{\underset{2}{\cancel{4}}}{\cancel{8}}\times\underset{1}{\cancel{4}}\times\underset{1}{\cancel{7}}}=\frac{81}{2}=40\tfrac{1}{2}.$$

The step in parentheses may be omitted.

$$\tfrac{16}{35}\times 1\tfrac{7}{8}\times\tfrac{21}{25}\times 3\tfrac{1}{3}\times 10\times 5\tfrac{5}{8}\times\tfrac{1}{7}$$

$$=\frac{\overset{\overset{1}{\cancel{2}}}{\cancel{16}}\times\overset{\overset{1}{\cancel{5}}}{\cancel{15}}\times\overset{3}{\cancel{21}}\times\overset{\overset{1}{\cancel{2}}}{\cancel{10}}\times\overset{\overset{1}{\cancel{2}}}{\cancel{10}}\times 45\times 1}{\underset{7}{\cancel{35}}\times\underset{1}{\cancel{8}}\times\underset{\underset{1}{\cancel{5}}}{\cancel{25}}\times\underset{1}{\cancel{3}}\times\underset{\underset{\underset{1}{\cancel{2}}}{\cancel{4}}}{\cancel{8}}\times\underset{1}{\cancel{7}}}=\frac{135}{7}=19\tfrac{2}{7}.$$

Exercise XXII.

(1) $\frac{1}{2}\times\frac{1}{3}$; $\frac{1}{7}\times\frac{1}{8}$; $\frac{1}{20}\times\frac{3}{5}$; $\frac{15}{16}\times\frac{18}{25}$; $\frac{4}{21}\times\frac{28}{29}$.

(2) $4\frac{1}{2}\times 6$; $4\frac{1}{2}\times 6\frac{1}{2}$; $4\frac{1}{3}\times 6\frac{1}{4}$; $100\frac{3}{8}\times 4\frac{4}{11}$.

(3) $7\frac{1}{4}\times\frac{18}{25}\times\frac{13}{58}\times\frac{8}{39}\times 5$; $4\frac{1}{2}\times 5\frac{1}{4}\times 5\frac{1}{3}\times 13$.

(4) $\frac{1}{10}\times\frac{1}{10}$; $\frac{3}{10}\times\frac{7}{10}$; $\frac{11}{100}\times\frac{7}{10000}$; $\frac{13}{100}\times\frac{13}{10000}\times\frac{13}{1000000}$.

(5) $5\frac{7}{10}\times 81\frac{13}{100}\times 4\frac{7}{1000}$; $7\frac{1}{10}\times 8\frac{1}{10}\times 9\frac{1}{10}$.

(6) $2\frac{1}{4}\times 20\frac{1}{4}\times\frac{26}{27}$; $\frac{5}{38}\times\frac{57}{65}\times 3\frac{1}{4}$; $\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$; $4\frac{1}{5}\times 4\frac{1}{5}\times 4\frac{1}{5}\times 4\frac{1}{5}$.

§ 6. $\frac{7}{8}\times\frac{8}{7}=1$. If the product of two numbers is unity, each is called the RECIPROCAL of the other. Thus 1 is the reciprocal of 1, 3 of $\frac{1}{3}$, $\frac{1}{3}$ of 3, $\frac{2}{3}$ of $1\frac{1}{2}$, $1\frac{1}{2}$ of $\frac{2}{3}$, &c.

Exercise M.

(1) Find the reciprocal of $\frac{5}{8}$. . . . . *Ans.* $1\frac{3}{5}$.
(2) „ $2\frac{1}{2}$. . . . . „ $\frac{2}{5}$.
(3) „ $6\frac{2}{3}$. . . . . „ $\frac{3}{20}$.
(4) „ $\frac{2}{9}$. . . . . „ $4\frac{1}{2}$.
(5) „ 7. . . . . „ $\frac{1}{7}$.
(6) „ $\frac{9}{10}$. . . . . „ $1\frac{1}{9}$.

(7) By what number must $\frac{4}{11}$ be multiplied to give for product 1 ? *Ans.* $2\frac{3}{4}$.

(8) By what must 5 be multiplied to give for product 1 ? „ $\frac{1}{5}$.

(9) By what must $6\frac{2}{3}$ be multiplied to give 1 ? „ $\frac{3}{20}$.

(10) Given product 1, one factor $\frac{5}{8}$, find the other factor. „ $1\frac{3}{5}$.

(11) By what must any number be multiplied to give 1 ? *Ans.* By its reciprocal.

(12) By what must a number be multiplied to give 2 ? *Ans.* By twice its reciprocal.

(13) By what must a number be multiplied to give 11 ? *Ans.* By 11 times its reciprocal.

(14) By what must a number be multiplied to give $\frac{1}{2}$? *Ans.* By $\frac{1}{2}$ of its reciprocal.

(15) By what must a number be multiplied to give $8\frac{2}{9}$? *Ans.* By $8\frac{2}{9}$ times its reciprocal.

(16) Write out ten pairs of reciprocals.

§ 7. DIVISION BY FRACTIONS.

$\frac{2}{3} \div \frac{9}{10}$. What does this mean ? In Part I. Ch. VIII. §§ 1 and 5, we have stated that the sign $\div$ bears *two* interpretations, either of which we may adopt if intelligible. In this case, *neither* is intelligible; for, 1st, $\frac{9}{10}$ being larger than $\frac{2}{3}$ cannot be contained in it; and 2nd, we cannot attach any idea to a distribution of $\frac{2}{3}$ into $\frac{9}{10}$ equal parts. A further extension of the meaning of the symbol $\div$ becomes necessary.

£15 ÷ £3 = 5 times, because $3 \times 5 = 15$.
£15 ÷ 3 = £5, because $3 \times 5 = 15$.

Thus we see that, under either interpretation, divisor × quotient = dividend. It follows that division means in each case; *Find the number by which the divisor must be multiplied to give the dividend.* This meaning applies to the question proposed, which is therefore, By what number must $\frac{9}{10}$ be multiplied to give $\frac{2}{3}$?

If we multiply it (viz. $\frac{9}{10}$) by $\frac{10}{9}$, we obtain unity; but we wish to obtain, not unity, but only $\frac{2}{3}$ of unity; we ought therefore to have multiplied only by $\frac{2}{3}$ of $\frac{10}{9}$. Hence, $\frac{2}{3} \div \frac{9}{10} = \frac{2}{3}$ of $\frac{10}{9} = \frac{2}{3} \times \frac{10}{9}$. We thus see that to divide by $\frac{9}{10}$ is equivalent to multiplying by its reciprocal, $\frac{10}{9}$. But this process of reasoning holds for any other

fractional divisor. Hence, generally: *To divide by a fraction, multiply by its reciprocal.* The same rule evidently holds also for integral divisors, for to divide any number by 3 is to take $\frac{1}{3}$ of it, i.e. to multiply it by $\frac{1}{3}$, and so on.

Learn by heart: *To divide by any number, multiply by its reciprocal.*

(*a*) $489 \div 7 = \frac{1}{7}$ of $489 = \frac{1}{7} \times 489 = \frac{489}{7} = 69\frac{6}{7}$.

(*b*) $\frac{4}{9} \div 7 = \frac{1}{7}$ of $\frac{4}{9} = \frac{4}{9} \times \frac{1}{7} = \frac{4}{63}$. (Cf. Part II. Ch. I. p. 14.)

(*c*) $\frac{3}{11} \div \frac{1}{11} = \frac{3}{11} \times 11 = 3$. (Cf. Part II. Ch. I. p. 16.)

(*d*) $\frac{7}{12} \div \frac{5}{8} = \frac{7}{\underset{3}{\cancel{12}}} \times \frac{\overset{2}{\cancel{8}}}{5} = \frac{14}{15}$. Verification: $\frac{\overset{7}{\cancel{14}}}{\underset{3}{\cancel{15}}} \times \frac{\overset{1}{\cancel{5}}}{\underset{4}{\cancel{8}}} = \frac{7}{12}$.

(*e*) $2\frac{5}{8} \div 2\frac{4}{5} = \frac{21}{8} \div \frac{14}{5} = \frac{\overset{3}{\cancel{21}}}{8} \times \frac{5}{\underset{2}{\cancel{14}}} = \frac{15}{16}$.

Verification: $\frac{15}{16} \times 2\frac{4}{5} = \frac{\overset{3}{\cancel{15}}}{\underset{8}{\cancel{16}}} \times \frac{\overset{7}{\cancel{14}}}{\underset{1}{\cancel{5}}} = \frac{21}{8} = 2\frac{5}{8}$.

We see now in its entirety that which indeed was dimly perceived before: (1st) that multiplication and division are opposite operations, by which we mean that they neutralize each other; (2nd) that they are interconvertible; every multiplication is a division by the reciprocal of the multiplier, and every division is a multiplication by the reciprocal of the divisor.

§ 8. We might have arrived at the same conclusion as follows:

$$\tfrac{2}{3} \div \tfrac{9}{10} = \tfrac{2}{3} \div \tfrac{1}{10} \text{ of } 9;\ \tfrac{2}{3} \div 9 = \frac{2}{3 \times 9} \text{ (Part II. Ch. I. p. 14)};$$

$\therefore\ \frac{2}{3} \div \frac{1}{10}$ of 9 ought to be 10 times as much, viz.

$$\frac{2}{3 \times 9} \times 10 = \frac{2 \times 10}{3 \times 9} \text{ (Ch. I. p. 11)} = \tfrac{2}{3} \times \tfrac{10}{9};$$

or thus:

$$\tfrac{2}{3} \div \tfrac{9}{10} = \frac{2 \times 10}{3 \times 10} \div \frac{9 \times 3}{10 \times 3} = 2 \times 10 \div 9 \times 3 = \frac{2 \times 10}{3 \times 9} = \tfrac{2}{3} \times \tfrac{10}{9}.$$

For beginners, the first demonstration is preferable, being notional rather than symbolic.

Exercise XXIII.

(1) $8 \div \frac{2}{3}$.
(2) $\frac{2}{3} \div 8$.
(3) $\frac{4}{5} \div \frac{3}{7}$.
(4) $\frac{3}{7} \div \frac{4}{5}$.
(5) $1\frac{4}{5} \div \frac{1}{5}$.
(6) $1\frac{4}{5} \div 5$.
(7) $1\frac{4}{5} \div 3$.
(8) $1\frac{4}{5} \div \frac{3}{5}$.
(9) $1\frac{4}{5} \div 1\frac{1}{8}$.
(10) $\frac{1}{5} \div 1\frac{4}{5}$.
(11) $\frac{3}{5} \div 1\frac{4}{5}$.
(12) $3 \div 1\frac{4}{5}$.
(13) $8\frac{4}{7} \div 4\frac{4}{9}$.
(14) $11 \div 76$.
(15) $11 \div 55$.
(16) $55 \div 11$.
(17) $1\frac{5}{7} \div \frac{2}{3}$.
(18) $1\frac{5}{7} \div 1\frac{2}{3}$.
(19) $5\frac{5}{8} \div 15$.
(20) $5\frac{5}{8} \div 3\frac{3}{4}$.
(21) $10 \div 2\frac{1}{2}$.
(22) $25 \div 3\frac{1}{8}$.
(23) $\frac{15}{16} \div 7\frac{1}{2}$.
(24) $\frac{11}{12} \div 1\frac{3}{8}$.
(25) $\frac{19}{20} \div 7\frac{3}{5}$.
(26) $\frac{25}{26} \div 1\frac{1}{4}$.
(27) $\frac{13}{15} \div 6\frac{1}{2}$.
(28) $19\frac{1}{3} \div \frac{29}{48}$.
(29) $5\frac{5}{6} \div \frac{21}{22}$.
(30) $18\frac{3}{4} \div 1\frac{7}{8}$.
(31) $9\frac{3}{8} \div 4\frac{2}{7}$.
(32) $100\frac{5}{9} \div 8\frac{4}{9}$.
(33) $23\frac{1}{3} \div 11\frac{2}{5}$.
(34) $588\frac{1}{8} \div 91\frac{1}{8}$.
(35) $1819\frac{3}{10} \div 81\frac{5}{7}$.
(36) $21 \div (\frac{15}{19}$ of $3\frac{4}{5})$.
(37) $18\frac{117}{256} \div (\frac{5}{8}$ of $33\frac{3}{4})$.
(38) $3\frac{1}{3} \div (\frac{2}{3} \times 1\frac{1}{3})$.
(39) $(3\frac{1}{11} \times 5\frac{1}{17}) \div 1720$.
(40) $(3\frac{1}{2} \times \frac{8}{9}$ of $\frac{4}{7}) \div 1\frac{3}{5}$.
(41) $(\frac{28}{59} \times \frac{59}{68}) \div (\frac{1}{6} \times 4 \times \frac{2}{3})$.
(42) $(13\frac{5}{8} \times \frac{2}{3} \times \frac{9}{65}) \div (\frac{2}{3}$ of $1\frac{44}{65} \times 1\frac{1}{8})$.
(43) £8. 7*s.* 10*d.* $\div 1\frac{2}{5}$.
(44) 13*s.* $8\frac{1}{4}$*d.* $\div 4\frac{7}{8}$.
(45) 4 days, 5 hours $\div 1\frac{5}{7}$.
(46) 3 qrs., 5 lbs., 8 oz. $\div 6\frac{2}{5}$.

(47) Find the cost of 1 article if $\frac{7}{10}$ cost 6*s.* 5*d.*
(48) If I earn 8*s.* 5*d.* in $1\frac{3}{5}$ days, how much is that a-day?
(49) If a soldier step $\frac{3}{4}$ of a yard, how many steps will he take in $1\frac{7}{8}$ miles?
(50) Divide $(\frac{17}{20} + \frac{11}{15} + \frac{7}{10} + \frac{4}{5})$ by $(\frac{17}{20} - \frac{11}{15} + \frac{7}{10} - \frac{4}{5})$.
(51) Divide $(4\frac{1}{7} - 2\frac{1}{4})$ by $(6\frac{1}{2} - 2\frac{1}{7})$.

Exercise XXIV.

(Miscellaneous Questions on the four Rules.)

(1) Find the sum of $4\frac{7}{12}$, $5\frac{2}{9}$, $7\frac{13}{16}$, and $10\frac{11}{20}$.
(2) What quantity exceeds $5\frac{2}{3}$ by $4\frac{7}{8}$?
(3) From what quantity must $6\frac{3}{5}$ be deducted to leave $\frac{1}{2}$ of $3\frac{1}{5}$?

(4) There are two fractions, the less is $10\frac{14}{25}$, their difference is $6\frac{8}{15}$. Find the greater.

(5) If from a certain quantity $2\frac{6}{7}$ be taken, $4\frac{11}{14}$ is left. Find the quantity.

(6) Of two weavers, A and B, A wove $9\frac{7}{10}$ pieces more than B, who wove $6\frac{11}{40}$ pieces. Find the total quantity woven.

(7) $5\frac{8}{11}$ exceeds a certain fraction by $(4 \div 2\frac{1}{5})$. Find the fraction.

(8) What fraction falls short of $\frac{7}{12}$ by $\frac{3}{20}$?

(9) What fraction is that to which $\frac{5}{76}$ must be added to give $\frac{11}{57}$?

(10) There are two fractions, the greater is $12\frac{7}{18}$, and their difference is $7\frac{53}{54}$. Find the less.

(11) What fraction increased by $\frac{1}{100}$ becomes $\frac{1}{10}$?

(12) Find a fraction which, repeated 3 times and increased by $14\frac{2}{5}$, makes 100.

(13) Find a fraction which, repeated 3 times and diminished by $14\frac{2}{5}$, makes 100.

(14) In a pair of scales, one contains $7\frac{4}{15}$ lbs., the other contains $11\frac{7}{9}$ lbs. Find the number of lbs. which drags down the heavier scale.

(15) Find the product of $4\frac{4}{9}$ and $3\frac{3}{5}$?

(16) What fraction must be divided by $7\frac{1}{2}$ to yield $7\frac{1}{2}$?

(17) From what number or fraction can $4\frac{17}{36}$ be taken 9 times exactly?

(18) From what fraction can $3\frac{5}{8}$ be taken $2\frac{1}{2}$ times, leaving remainder $3\frac{1}{2}$?

(19) Of what fraction is $10\frac{14}{15}$ the 10th part?

(20) What fraction divided by $4\frac{5}{7}$ gives the quotient $\frac{2}{3}$?

(21) What is the 7th part of $1\frac{1}{2}$?

(22) What fraction is that of which we must take $6\frac{2}{3}$ to get $5\frac{1}{2}$?

(23) By what fraction must 10 be multiplied to give 7?

(24) The product of two fractions is $\frac{5}{8}$; one factor is $1\frac{1}{4}$. Find the other.

(25) Given divisor $3\frac{1}{5}$, quotient $3\frac{1}{5}$. Find dividend.

(26) Given dividend $\frac{11}{12}$, quotient $6\frac{1}{2}$. Find divisor.

(27) Given dividend $\frac{11}{12}$, divisor $6\frac{1}{2}$. Find quotient.

(28) Given dividend $14\frac{2}{5}$, quotient $3\frac{3}{5}$, remainder $1\frac{5}{12}$. Find divisor.

§ 9. LITERAL SUMMARY OF THE RULES ON FRACTIONS.

Let a, b, c, d, m, &c., represent any integers whatever.

Formulæ.	*Examples.*
(α) $\frac{a}{b}=(\frac{1}{b}\text{ of }1)\times a=\frac{1}{b}\text{ of }a.$ (Ch. I. § 3, Ch. II. § 2.)	$\frac{2}{3}=(\frac{1}{3}\text{ of }1)\times 2=\frac{1}{3}\text{ of }2$
(β)* If $a<b$, $\frac{a}{b}<1.$	$\frac{7}{12}<1$
If $a=b$, $\frac{a}{b}=1.$	$\frac{12}{12}=1$
† If $a>b$, $\frac{a}{b}>1.$ (Ch. I. § 4.)	$\frac{13}{12}>1$
(γ) $a+\frac{b}{c}=\frac{a\times c+b}{c}$ (Ch. I. § 5.)	$4\frac{5}{7}=\frac{4\times 7+5}{7}=\frac{33}{7}$
(δ) $\frac{a}{b}=a\div b.$ (Ch. I. § 6, Ch. II. § 2.)	$\frac{23}{5}=23\div 5=4\frac{3}{5}$
(ϵ) $\frac{a}{d}+\frac{b}{d}+\frac{c}{d}=\frac{a+b+c}{d}$	$\frac{4}{19}+\frac{6}{19}+\frac{7}{19}=\frac{4+6+7}{19}=\frac{17}{19}$
$\frac{a}{d}-\frac{b}{d}=\frac{a-b}{d}$ (Ch. I. § 7.)	$\frac{17}{19}-\frac{12}{19}=\frac{17-12}{19}=\frac{5}{19}$
(ζ) $\frac{a}{b}\times m=\frac{a\times m}{b}=\frac{a}{b\div m}$ (Ch. I. § 8.)	$\frac{7}{18}\times 6=\frac{7\times 6}{18}=\frac{42}{18}=2\frac{6}{18}$; or, $\frac{7}{18}\times 6=\frac{7}{18\div 6}=\frac{7}{3}=2\frac{1}{3}$
(η) $\frac{a}{b}\div m=\frac{a\div m}{b}=\frac{a}{b\times m}$ (Ch. I. § 9.)	$\frac{12}{13}\div 4=\frac{12\div 4}{13}=\frac{3}{13}$; or $=\frac{12}{13\times 4}=\frac{12}{52}$
(θ) $\frac{a}{b}=\frac{a\times m}{b\times m}=\frac{a\div m}{b\div m}$ (Ch. II. § 1.)	$\frac{20}{25}=\frac{20\times 3}{25\times 3}=\frac{60}{75}$; or $=\frac{20\div 5}{25\div 5}=\frac{4}{5}$
(ι) $\frac{1}{a}\text{ of }\frac{1}{b}=\frac{1}{a\times b}$ (Ch. II. § 5.)	$\frac{1}{4}\text{ of }\frac{1}{5}=\frac{1}{4\times 5}=\frac{1}{20}$
(κ) $\frac{a}{b}\text{ of }\frac{c}{d}=\frac{a\times c}{b\times d}$ (Ch. II. § 6.)	$\frac{5}{7}\text{ of }\frac{3}{8}=\frac{5\times 3}{7\times 8}=\frac{15}{56}$

* Read, "If a is less than b."

† Read, "If a is greater than b."

(6) What fraction of 15s. 4d. is £8. 10s. 10d.?
(7) „ £3. 2s. $7\frac{1}{4}d.$ is £1?
(8) „ £1 is £3. 2s. $7\frac{1}{4}d.$?
(9) „ 2 tons, 13 cwt., is 1 ton, 1 cwt., 1 qr.?
(10) „ 2 lbs., 10 oz. av., is 1 lb., $4\frac{1}{2}$ oz. av.?
(11) „ 5 lbs., 9 oz. troy, is 6 oz., 15 dwts.?
(12) „ 2 yrs., 73 days, is 146 days?
(13) „ 5 tons, 8 cwt., 21 lbs., is 3 cwt., 17 lbs., 12 oz.?
(14) Reduce £2. 10s. 6d. to the fraction of £1. 10s. 9d.
(15) „ 6s. 7d. „ £8.
(16) „ 2 lbs., 10 oz. av. „ 2 lbs., 10 oz. troy.
(17) „ 5 minutes „ 1 day.
(18) „ $1\frac{1}{2}d.$ „ 1 guinea.
(19) „ $1\frac{1}{2}$ pints „ $2\frac{1}{2}$ gallons.
(20) „ 1 lb. av. „ 1 lb. troy.

§ 2. INTERPRETATION OF FRACTIONS.

We have found that the symbol $\frac{3}{4}$ bears the following interpretations:

(α) A quarter of *one* thing taken *three* times.

(β) A quarter of *three* things taken *once*.

(γ) $3 \div 4$.

(δ) The fraction that 3 is of 4.

(ϵ) The number by which 4 must be multiplied to give 3.

These are but different modes of expression for the same notion, and any one may be selected according to convenience. If the numerator and denominator are both abstract integers, they are all immediately intelligible.

Examine the symbol $\frac{4\frac{1}{5}}{2\frac{1}{3}}$. The interpretations ($\gamma$) ($\delta$) and ($\epsilon$) are readily intelligible, but (α) and (β) require an extension of language.

(α) may be given in these words: "Of equal pieces such that 4 of them make the whole, take 3." This wording applies to $\frac{4\frac{1}{5}}{2\frac{1}{3}}$, substituting $2\frac{1}{3}$ and $4\frac{1}{5}$ for 4 and 3 respectively.

(β) may be given thus: "Of equal pieces such that 4 of them make 3 wholes, take 1." This wording also applies to $\frac{4\frac{1}{5}}{2\frac{1}{3}}$ by making the above substitution.

We proceed to examine whether these five interpretations will all, when applied to the symbol $\frac{4\frac{1}{5}}{2\frac{1}{3}}$, yield the same result.

(α) If $2\frac{1}{3}$ pieces make a unit, each piece must be $\frac{3}{7}$ of 1; for $2\frac{1}{3}$ of $\frac{3}{7} = \frac{7}{3}$ of $\frac{3}{7} = 1$. If now we take $4\frac{1}{5}$ of such pieces, we obtain $4\frac{1}{5} \times \frac{3}{7} = \frac{21}{5} \times \frac{3}{7} = \frac{9}{5} = 1\frac{4}{5}$. *Ans.* $1\frac{4}{5}$.

(β) If $2\frac{1}{3}$ pieces make $4\frac{1}{5}$ wholes, each piece must be $4\frac{1}{5}$ times as great as if $2\frac{1}{3}$ pieces made 1; that is, $4\frac{1}{5} \times \frac{3}{7} = 1\frac{4}{5}$ as above. *Ans.* $1\frac{4}{5}$.

(γ) $4\frac{1}{5} \div 2\frac{1}{3} = 4\frac{1}{5} \times \frac{3}{7} = 1\frac{4}{5}$. *Ans.* $1\frac{4}{5}$.

(δ) What fraction of $2\frac{1}{3}$ is $4\frac{1}{5}$? 1 is $\frac{3}{7}$ of $2\frac{1}{3}$, $\therefore$ $4\frac{1}{5}$ is $(4\frac{1}{5} \times \frac{3}{7})$ of $2\frac{1}{3}$; $4\frac{1}{5} \times \frac{3}{7} = 1\frac{4}{5}$. *Ans.* $1\frac{4}{5}$.

(ϵ) By what number must $2\frac{1}{3}$ be multiplied to give $4\frac{1}{5}$? (Ch. III. § 7.) $4\frac{1}{5} \div 2\frac{1}{3} = 1\frac{4}{5}$. *Ans.* $1\frac{4}{5}$.

We thus see that the interpretation chosen will not affect the result, whether the terms of the fraction be integral or fractional. The interpretation (γ) is of easiest application, and therefore most generally adopted.

§ 3. Simplification of Fractions.

By simplification of a fraction, is meant finding the simplest possible expression whose value is equal to that of the given fraction. Hence the answer to a simplification of fractions ought always to be either an integer, a proper fraction at lowest terms, or a mixed number.

Simplify $\frac{8\frac{4}{7}}{1\frac{19}{21}}$.

$$\frac{8\frac{4}{7}}{1\frac{19}{21}} = 8\frac{4}{7} \div 1\frac{19}{21} = \frac{60}{7} \div \frac{40}{21} = \frac{\overset{3}{\cancel{60}}}{\underset{1}{\cancel{7}}} \times \frac{\overset{3}{\cancel{21}}}{\underset{2}{\cancel{40}}} = \frac{9}{2} = 4\frac{1}{2}. \qquad \textit{Ans. } 4\tfrac{1}{2}.$$

Simplify $\frac{\frac{5}{7} + \frac{1}{4}}{2\frac{1}{4} - 1\frac{5}{7}}$.

$$\frac{\frac{5}{7} + \frac{1}{4}}{2\frac{1}{4} - 1\frac{5}{7}} = \frac{\frac{20}{28} + \frac{7}{28}}{2\frac{7}{28} - 1\frac{20}{28}} = \frac{\frac{27}{28}}{\frac{9}{14}} = \frac{27}{28} \div \frac{9}{14} = \frac{\overset{3}{\cancel{27}}}{\underset{2}{\cancel{28}}} \times \frac{\overset{1}{\cancel{14}}}{\underset{1}{\cancel{9}}} = \frac{3}{2} = 1\frac{1}{2}. \quad \textit{Ans. } 1\tfrac{1}{2}.$$

Simplify $\frac{\frac{14}{27} \times \frac{18}{35}}{1\frac{23}{25} \text{ of } \frac{5}{36}}$.

$$\frac{\frac{14}{27} \times \frac{18}{35}}{1\frac{23}{25} \text{ of } \frac{5}{36}} = \frac{\overset{1}{\overset{\cancel{2}}{\cancel{14}}}}{\underset{1}{\underset{\cancel{9}}{\cancel{27}}}} \times \frac{\overset{1}{\overset{\cancel{9}}{\cancel{18}}}}{\underset{1}{\underset{\cancel{7}}{\cancel{35}}}} \times \frac{\overset{1}{\overset{\cancel{5}}{\cancel{25}}}}{\underset{1}{\underset{\cancel{2}}{\underset{8}{\cancel{48}}}}} \times \frac{\overset{1}{\overset{\cancel{4}}{\cancel{36}}}}{\underset{1}{\cancel{5}}} = 1. \qquad \textit{Ans. } 1.$$

Simplify $\dfrac{1}{6+\frac{1}{7\frac{1}{8}}}$

$\frac{1}{7\frac{1}{8}}=\frac{8}{57}$; $6+\frac{8}{57}=\frac{350}{57}$; $\frac{1}{\frac{350}{57}}=\frac{57}{350}$. *Ans.* $\frac{57}{350}$.

Simplify $\dfrac{2\frac{3}{8}}{3\frac{1}{6}}$ of $\dfrac{4\frac{4}{9}}{\frac{1}{16}} \div \dfrac{8\frac{8}{9}}{7}$ of $\dfrac{4}{5\frac{5}{8}}$.

$$\frac{2\frac{3}{8}\times 4\frac{4}{9}\times 7\times 5\frac{5}{8}}{3\frac{1}{6}\times\frac{1}{16}\times 8\frac{8}{9}\times 4}=\frac{\overset{1}{\cancel{19}}\times\overset{3}{\cancel{6}}\times\overset{1}{\cancel{40}}\times\overset{1}{\cancel{16}}\times\overset{1}{\cancel{9}}\times 7\times 45}{\underset{4}{\cancel{8}}\times\underset{1}{\cancel{19}}\times\underset{1}{\cancel{9}}\times\underset{\underset{1}{\cancel{2}}}{\cancel{80}}\times 4\times\underset{1}{\cancel{8}}}=\frac{945}{16}=59\tfrac{1}{16}.\quad \textit{Ans. } 59\tfrac{1}{16}.$$

Simplify $\dfrac{(\frac{4}{11}\div\frac{3}{5})+\frac{1}{11}\text{ of }2+\frac{1}{6}\times(\frac{1}{5}-\frac{1}{11})}{\frac{1}{4\frac{1}{2}}+\frac{1}{5\frac{1}{2}}+\frac{1}{7\frac{1}{2}}}$.

$\frac{4}{11}\div\frac{3}{5}=\frac{20}{33}$; $\frac{1}{11}$ of $2=\frac{2}{11}$; $\frac{1}{6}$ of $(\frac{1}{5}-\frac{1}{11})=\frac{1}{6}$ of $\frac{6}{55}=\frac{1}{55}$.

$$\frac{20}{33}+\frac{2}{11}+\frac{1}{55}=\frac{100+30+3}{165}=\frac{133}{165}.$$

$\frac{1}{4\frac{1}{2}}=1\div\frac{9}{2}=\frac{2}{9}$; $\frac{1}{5\frac{1}{2}}=\frac{2}{11}$; $\frac{1}{7\frac{1}{2}}=\frac{2}{15}$.

$$\frac{2}{9}+\frac{2}{11}+\frac{2}{15}=\frac{110+90+66}{495}=\frac{266}{495}.$$

$$\frac{133}{165}\div\frac{266}{495}=\frac{\overset{1}{\cancel{133}}\times\overset{\overset{3}{\cancel{99}}}{\cancel{495}}}{\underset{\underset{1}{\cancel{33}}}{\cancel{165}}\times\underset{2}{\cancel{266}}}=\frac{3}{2}=1\tfrac{1}{2}.\qquad \textit{Ans. } 1\tfrac{1}{2}.$$

EXERCISE XXVII.

(1) $\dfrac{1}{3\frac{1}{5}}$

(2) $\dfrac{1}{3\frac{2}{5}}$

(3) $\dfrac{5}{3\frac{2}{5}}$

(4) $\dfrac{5\frac{1}{2}}{3\frac{2}{5}}$

(5) $\dfrac{3\frac{2}{5}}{5\frac{1}{2}}$

(6) $\dfrac{1\frac{1}{2}}{3\frac{7}{16}}$

(7) $\dfrac{4\frac{8}{27}}{2\frac{14}{81}}$

(8) $\dfrac{5\frac{13}{28}}{2\frac{19}{49}}$

(9) $\dfrac{2\frac{5}{8}}{8\frac{2}{5}}\times\dfrac{8\frac{2}{5}}{2\frac{5}{8}}$

(10) $\dfrac{2\frac{5}{8}\times 8\frac{2}{5}}{8\frac{2}{5}\div 2\frac{5}{8}}$

(11) $\dfrac{2\frac{5}{8}+8\frac{2}{5}}{8\frac{2}{5}-2\frac{5}{8}}$

(12) $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\frac{1}{2\frac{1}{2}}+\frac{1}{3\frac{1}{2}}+\frac{1}{4\frac{1}{2}}}$

(13) $\dfrac{\frac{7}{8}+\frac{4}{9}+\frac{1}{10}}{\frac{1}{6}+\frac{1}{45}+\frac{1}{72}}$

(14) $\dfrac{\frac{7}{8}-(\frac{4}{9}+\frac{1}{10})}{\frac{1}{6}+\frac{1}{45}-\frac{1}{72}}$

(15) $\dfrac{1\frac{4}{17}\times 6\frac{4}{5}}{3\frac{3}{5}-1\frac{2}{25}}+\frac{2}{11}$ of $(2\frac{2}{5}-\frac{3}{4})-\dfrac{1\frac{3}{5}}{12}$

(16) $3\frac{1}{3}\div\dfrac{1-\frac{15}{16}}{\frac{1}{5}-\frac{1}{8}}$

(17) $\cfrac{1}{2+\cfrac{1}{3\frac{1}{2}-\frac{7}{8}}}$

(18) $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}$

(19) $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}$

(20) $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}$

(21) $\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{15+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{2}}}}}}$

(22) $3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{25+\cfrac{1}{1+\cfrac{1}{7+\cfrac{1}{4}}}}}}}$

(23) $\dfrac{\frac{1}{3}+\frac{4}{9}+\frac{11}{27}}{\frac{1}{4}+\frac{5}{6}+\frac{11}{12}}$ of $\dfrac{\frac{5}{8} \text{ of } 2\frac{1}{3}\div 1\frac{11}{24}}{\frac{16}{27} \text{ of } 4\frac{5}{8}\div 9\frac{1}{4}}$

§ 4. G.C.M. AND L.C.M. OF FRACTIONS.

The meaning of the expressions Measure and Multiple requires to be rendered somewhat more precise. If of two quantities the first is contained in the second an *integral* number of times, the first is a measure of the second, and the second a multiple of the first.

Examine $\frac{8}{3}\div\frac{2}{9}$. $\frac{8}{3}\div\frac{2}{9}=\frac{8}{3}\times\frac{9}{2}$.

$$\frac{\overset{4}{\cancel{8}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{2}}}=12.$$

This quotient is integral only because the 2 is a measure of the 8 and the 3 of the 9, as otherwise the denominators would not disappear. Generally: If $\frac{a}{b}$ and $\frac{c}{d}$ are two fractions at their lowest terms, $\frac{a}{b}\div\frac{c}{d}$ is integral only if c is a measure of a, and d a multiple of b.

In order, therefore, that one quantity should measure a series of quantities (of course at their lowest terms), its numerator must be a measure of each numerator, and its denominator a multiple of each denominator of the series. Hence G.C.M. of a series of fractions = G.C.M. of numerators $\div$ L.C.M. of denominators; and conversely, L.C.M. of a series = L.C.M. of numerators $\div$ G.C.M. of denominators. We may

also remark that the resulting G. C. M. or L. C. M. will be at its lowest terms; for if the G. C. M. of the numerators, say, have any factor, and that factor is contained in any one of the original denominators, one at least of the given fractions would not be at lowest terms.

Find G. C. M. and L. C. M. of $\frac{9}{35}$, $\frac{12}{25}$, $\frac{27}{50}$.

G. C. M. of 9, 12, 27, is 3. G. C. M. of 35, 25, 50, is 5.
L. C. M. of „ is 108. L. C. M. of „ is 350.

∴ G. C. M. required is $\frac{3}{350}$, and L. C. M. required is $\frac{108}{5} = 21\frac{3}{5}$.

Exercise XXVIII.

(1) Find G. C. M. and L. C. M. of $5\frac{1}{2}$, $7\frac{1}{3}$, $8\frac{1}{4}$, $4\frac{8}{9}$, $9\frac{1}{6}$, $6\frac{5}{12}$.
(2) „ „ $33\frac{3}{7}$ and $50\frac{5}{8}$.
(3) „ „ $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{12}$.
(4) „ „ $50\frac{1}{2}$, $67\frac{1}{3}$, $44\frac{8}{9}$, $84\frac{1}{6}$, 707.
(5) „ „ $225\frac{3}{5}$ and $181\frac{2}{7}$.
(6) „ „ $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$.
(7) „ „ $1\frac{1}{14}$, $1\frac{19}{21}$, $4\frac{2}{7}$, $\frac{25}{42}$.

§ 5. Surface Measure.

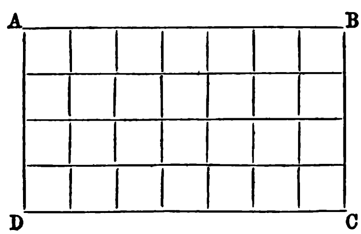
The extent of a surface expressed numerically is called its *area.*

A square surface which measures an inch each way is called a *square inch.*

A square surface which measures a foot each way is called a *square foot.* Similarly we have square yards, square miles, &c.

The area of a surface is expressed by the number of such square units that it contains,* or that would be required to cover it.

Find the area of a table 7 feet long and 4 feet broad.



* It will be seen that we only deal here with rectangular plane surfaces.

Let the line AB represent the length, which contains 7 *linear* feet, and AD the breadth, containing 4 linear feet. If lines be drawn as in the figure, the space ABCD will be divided into *square* feet, and there will be 4 rows of 7 square feet each, or 7 columns of 4 square feet each, i. e. $7 \times 4 = 28$ square feet.

Similar reasoning will in every case shew that the number of units in one side multiplied by the number of units in the other, will give the number of square units in the area.

Let l =	the number of units		in the	length.
„ b =	„		„	breadth.
„ a =	„	square units	„	area.

Then $l \times b = a$.

Exercise XXIX.

	From length,			breadth,			find area :
(1)	15	linear	feet,	7	linear	feet.	
(2)	10	„	yds.,	3	„	yds.	
(3)	5	„	in.,	2	„	in.	
(4)	1	„	foot,	1	„	in.	
(5)	2	„	yds.,	4	„	feet.	
(6)	12	„	in.,	12	„	in.	
(7)	3	„	feet,	3	„	feet.	
(8)	1760	„	yds.,	1760	„	yds.	

The following table will now be evident:

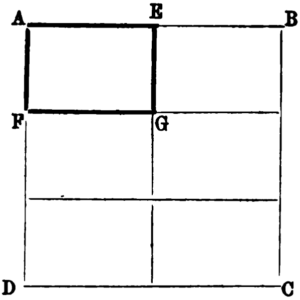
144 square inches = 1 square foot.
9 square feet = 1 square yard.

Since $a = l \times b$, if a and l are known, b can be found; for the question is, By what number must l be multiplied to give a? *Ans.* $a \div l$, or $\frac{a}{l}$. Hence, $\frac{a}{l} = b$; similarly, $\frac{a}{b} = l$.

Exercise XXX.

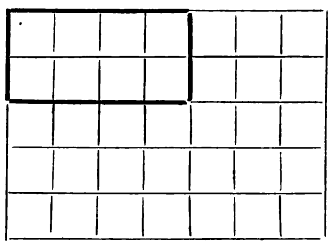
	From (or) Given } Area,			and one side,			find other side :
(1)	36	square	yds.,	9	linear	yds.	
(2)	28	„	in.,	7	„	in.	
(3)	100	„	feet,	5	„	feet.	
(4)	100	„	miles,	10	„	miles.	

Find the area if the length is $\frac{1}{2}$ ft. and the breadth $\frac{1}{3}$ ft.



Let AB = 1 foot, AE = $\frac{1}{2}$ ft., AD = 1 foot, AF = $\frac{1}{3}$ ft., then ABCD is 1 square foot, and AEGF is $\frac{1}{6}$ of 1 square foot, i.e. $\frac{1}{2} \times \frac{1}{3}$ square ft. Similar reasoning will shew in every case that if the length is $\frac{1}{a}$ and breadth $\frac{1}{b}$, the area is $\frac{1}{a} \times \frac{1}{b}$.

Next suppose length $\frac{4}{7}$ ft., breadth $\frac{2}{5}$ ft.



Examination of the diagram shews that, as before, length $\frac{1}{7}$ ft., breadth $\frac{1}{5}$ ft., gives $\frac{1}{35}$ sq. ft.; but length $\frac{4}{7}$, breadth $\frac{2}{5}$, gives $4 \times 2 \times \frac{1}{35} = \frac{8}{35}$, i.e. $\frac{4}{7} \times \frac{2}{5}$. Hence in every case, whether the given dimensions be fractional or integral, $l \times b = a$, and consequently $\frac{a}{b} = l$, and $\frac{a}{l} = b$.

Find the area if the length is 5 ft., 4 in., and breadth 2 ft., 9 in.

5 ft., 4 in. $= 5\frac{1}{3}$ ft.; 2 ft., 9 in. $= 2\frac{3}{4}$ ft.

$$5\tfrac{1}{3} \times 2\tfrac{3}{4} = \frac{\overset{4}{\cancel{16}}}{3} \times \frac{11}{\underset{1}{\cancel{4}}} = \frac{44}{3} = 14\tfrac{2}{3} \text{ sq. ft.} = 14 \text{ sq. ft., } 96 \text{ sq. in.}$$

Or, 5 ft., 4 in. = 64 in.; 2 ft., 9 in. = 33 in.

$64 \times 33 = 2112$ sq. in.

$$\begin{array}{r|l} 12 & \underline{2112} \\ 12 & \underline{176} \\ & 14 \text{ sq. ft. and } 8 \times 12 \text{ sq. in.} \end{array}$$

Ans. 14 sq. ft., 96 sq. in.

Required the length of carpet $\frac{3}{4}$ yd. wide, to cover a room 20 ft., 10 in. long, 15 ft., $10\frac{1}{2}$ in. broad.

The area of the floor is $\frac{20\frac{5}{6}}{3} \times \frac{15\frac{7}{8}}{3}$ sq. yds. The area of the carpet is the same; and as its width is known, the length is found by dividing the area by the width, $\frac{3}{4}$ yd.

$$\frac{20\frac{5}{6}}{3} \times \frac{15\frac{7}{8}}{3} \times \frac{4}{3} = \frac{125}{6\times3} \times \frac{127}{\underset{2}{\cancel{8}}\times3} \times \frac{\overset{1}{\cancel{4}}}{3} = \frac{15875}{324} = 48\tfrac{323}{324} \text{ yds.}$$

Exercise XXXI.

(1) Find the area, given:

a.	Length,	1 yd., 2 ft., 9 in.	breadth,	2 ft., 8 in.
b.	„	2 yds., 1 ft., $7\frac{1}{2}$ in.	„	$10\frac{1}{2}$ in.
c.	„	17 yds., 2 ft., 3 in.	„	1 yd., 1 ft., 10 in.
d.	„	10 yds., 2 ft., 11 in.	„	1 yd.
e.	„	1 ft., $1\frac{1}{2}$ in.	„	1 ft.
f.	„	36 yds.	„	4 yds., 1 ft., 5 in.
g.	„	$\frac{7}{11}$ yd.	„	$\frac{3}{8}$ yd.
h.	„	$\frac{7}{11}$ yd.	„	$\frac{3}{8}$ ft.
k.	„	$2\frac{5}{9}$ ft.	„	$5\frac{3}{8}$ in.
l.	„	$\frac{1}{2}$ in.	„	$\frac{1}{2}$ in.

(2) A room is 20 ft. long, 14 ft., 6 in. broad, and 11 ft., 9 in. high. How many square feet of carpet will be required, and how many of paper?

(3) What length of carpet, 2 ft., 3 in. wide, and of paper, 1 ft., 9 in. wide, will be required?

(4) How many acres in a square furlong?

(5) A rod or pole is $5\frac{1}{2}$ yds. How many square yards in a square rod?

(6) Express in acres the difference between half a square mile and half a mile square.

(7) Find the difference between 2 square furlongs and 2 furlongs square.

(8) The difference between $3\frac{1}{8}$ square yards and $3\frac{1}{8}$ yards square.

(9) Find the one side, given:

a.	Area,	50 square ft.	the other side,	$3\frac{1}{2}$ yds.
b.	„	$185\frac{2}{3}$ square ft.	„	4 yds., 2 ft., $7\frac{1}{2}$ in.
c.	„	$6\frac{1}{2}$ square ft.	„	$10\frac{2}{5}$ ft.

(10) How many pieces of paper, 12 yds. long and 21 in. wide, will paper a room $5\frac{1}{4}$ yds. wide, $7\frac{1}{2}$ yds. long, $8\frac{3}{4}$ ft. high, and what will be the cost at 2*s.* $10\frac{1}{2}$*d.* per piece?

(11) Find the difference in expense between carpeting a room 18 ft. long, 13 ft., 8 in. wide, with Brussels, 27 in. wide, at 5*s.* 6*d.* a yard, and with Kidderminster a yard wide at 3*s.* 9*d.* a yard.

(12) Find the cost of 9 venetian blinds, 7 ft., 10 in. long, and 4 ft. $7\frac{1}{2}$ in. wide, at $8\frac{1}{2}$*d.* per square foot.

§ 6. SOLID MEASURE.

The space filled by a body, expressed numerically, is called its *volume*, or cubic content.

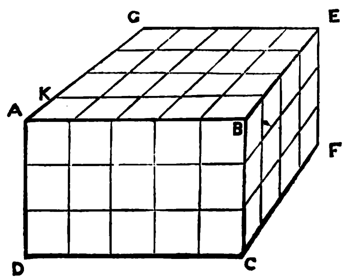
A cube which measures an inch each way is called a *cubic inch.*

A cube which measures a foot each way is called a *cubic foot.*

Similarly we have cubic yards, &c.

The volume of a body is expressed by the number of such cubic units that it contains, or that would fill the same amount of space.

Find the volume of a block of marble, 5 ft. long, 4 ft. broad, and 3 ft. thick.



Let AB be the length, 5 feet; AD the thickness, 3 feet; AG the breadth, 4 feet.

The area of the face, $ABCD = 5 \times 3 = 15$ square feet. If the breadth were 1 foot, as AK, the block would yield 15 cubic feet; but being 4 feet broad, the block yields $4 \times 15 = 60$ cubic feet.

Similar reasoning will in every case shew that the number of square units in one face must be multiplied by the number of linear units in the side independent of that face. Let l, b, t, be the number of units in the length, breadth and thickness respectively; $l \times b$, $b \times t$, $t \times l$, are the areas of the several faces; and $l \times b \times t = v$, v being the volume.

The following table will now be evident:

1728 cubic inches = 1 cubic foot.
27 cubic feet = 1 cubic yard.

Exercise XXXII.

From length,	breadth,	thickness,	find volume:
(1) 8 ft.	5 ft.	3 ft.	
(2) 10 in.	10 in.	10 in.	
(3) 1 yd.	1 ft.	1 in.	
(4) $2\frac{1}{2}$ yds.	$1\frac{3}{4}$ yds.	$\frac{5}{8}$ yds.	
(5) 10 yds., 1 ft.	2 ft., 8 in.	1 ft., $10\frac{1}{2}$ in.	
(6) 1 mile	15 yds.	$10\frac{1}{2}$ ft.	

Since $v = l \times b \times t$, $\frac{v}{l \times b} = t$; $\frac{v}{l \times t} = b$; $\frac{v}{t \times b} = l$; $\frac{v}{l} = b \times t$; $\frac{v}{b} = t \times l$; $\frac{v}{t} = l \times b$.

Exercise XXXIII.

(1) Given volume,	length,	and breadth, find thickness:
a. 120 cub. ft.	8 ft.	5 ft.
b. 5760 cub. in.	1 yd.	1 ft., 8 in.
c. 9 cub. yds., 7 ft., 1512 in.	6 yds., 7 in.	2 ft., 3 in.
d. 1 cub. yd.	4 yds.	2 yds., 1 ft.
e. $4\frac{7}{8}$ cub. ft.	$3\frac{1}{3}$ yds.	$2\frac{1}{4}$ in.

(2) Given volume,	area of one face, find the third side:
a. 60 cub. ft.	12 sq. ft.
b. 5 cub. yds., 25 ft.	17 sq. yds., 7 ft.
c. $8\frac{1}{15}$ cub. yds.	$1\frac{2}{9}$ sq. ft.
d. $2\frac{1}{4}$ cub. in.	150 sq. yds.

(3) Given volume and one side, find area of one face:

a. 343 cub. yds.	7 yds.
b. 1 cub. yd.	$1\frac{1}{2}$ in.
c. $3\frac{3}{8}$ cub. in.	$\frac{1}{1000}$ in.
d. $1\frac{2}{5}$ cub. ft.	1 yd.

(4) How many bricks are there in a wall, 7 yds., 2 ft., 4 in. long, 2 yds., 2 ft., 3 in. high, 1 ft., $1\frac{1}{2}$ in. thick? A brick is 9 in. long, $4\frac{1}{2}$ in. broad, 3 in. thick.

(5) A pile of stones is 12 yds. long, $4\frac{1}{2}$ yds. broad, 5 ft. high. How many stones are there, each being 1 foot long, $4\frac{1}{2}$ in. broad, 5 in. thick?

(6) How many cubic feet of wood are there in a block of timber, 15 ft., $10\frac{1}{2}$ in. long, 2 ft., $7\frac{1}{2}$ in. broad, and 1 ft., 2 in. thick?

(7) Gold can be beaten out to the thickness of $\frac{1}{200000}$ of an inch. How much surface can a cubic inch of gold be made to cover?

(8) If $\frac{2}{5}$ of a cubic foot of gold is beaten out to cover 1 acre, find its thickness.

(9) Find the value of a mass of timber, 6 yds., 2 ft., 4 in. long, 1 ft., 9 in. broad, 10 in. thick, at $7\frac{3}{4}d.$ per cubic foot.

(10) How large must the reservoir be that would contain the daily water supply of London, 83361824 gallons, if each gallon has $277\frac{11}{40}$ cubic inches, supposing it to be 6 feet deep?

(11) Find the cost of lining with tin a cubical box, one edge of which is 4 ft., 6 in., at 1*s.* 6*d.* per square yard.

Exercise XXXIV.

Miscellaneous Examples on Vulgar Fractions.

(1) Find the value of:

a. $\frac{15}{28}$ of £2. 13*s.* $1\frac{1}{4}d.$

b. $\frac{15}{28}$ of 7 cwt., 3 qrs., 9 lbs.

c. $\frac{15}{28}$ of 2 years, 73 days.

(2) Find the value of $4\frac{4}{9}$ guineas + $3\frac{2}{5}$ of 2*s.* 6*d.* + £$\frac{7}{15}$ − $\frac{3}{5}$ of 1*s.*

(3) Simplify $(7\frac{5}{12} + 6\frac{5}{8}) + (7\frac{5}{12} - 6\frac{5}{8}) + (7\frac{5}{12} \text{ of } 6\frac{5}{8}) - (7\frac{5}{12} \times 6\frac{5}{8}) + [(7\frac{5}{12} \div 6\frac{5}{8}) \times (6\frac{5}{8} \div 7\frac{5}{12})]$.

(4) Interpret the symbol $\times$, and apply it, where possible, to the following:

a. £7 × 3.	*c.* £7 × 0.	*e.* £$\frac{3}{7} \times \frac{5}{9}$.	*g.* $\frac{3}{7} \times$ £$\frac{5}{9}$.
b. £7 × 1.	*d.* £$\frac{2}{5} \times \frac{1}{2}$.	*f.* £$\frac{5}{9} \times \frac{3}{7}$.	*h.* £$\frac{3}{7} \times$ £$\frac{5}{9}$.

(5) What fraction of £1 are 13*s.* 8*d.*, 17*s.* $10\frac{1}{2}$*d.*, 2*s.* $3\frac{1}{4}$*d.*?

(6) Divide $5\frac{3}{8}$ of 2*s.* 6*d.* by $\frac{3}{7}$ of 1*s.*

(7) A can do a piece of work in 4 days, and B in 5 days. In what time will they do it together?

[A in 1 day does $\frac{1}{4}$ of the work;
B " $\frac{1}{5}$ of the work;
∴ A and B together " do $\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$ of the work; as many times, then, as $\frac{9}{20}$ of the work is contained in the whole work, so many days will A and B together require. $1 \div \frac{9}{20} = \frac{20}{9} = 2\frac{2}{9}$.]

(8) A can do a piece of work in 12 days, which B can do in 10 and C in 15 days. In what time will all three together do it?

(9) A can do a piece of work in 14 days; B works twice as fast as A, and C can do it in 10 days. In what time can all three together do it?

(10) A can mow 2 acres in 7 days, B can mow 3 acres in 10 days. In what time will the two together mow 1 acre?

(11) A and B together can do a piece of work in 8 days, A alone can do it in 12 days. In what time could B alone do it?

(12) Simplify $\dfrac{\frac{2\frac{1}{2}}{1} + 1\frac{2}{3} + \frac{1}{3\frac{1}{2}}}{3\frac{2}{3} - 2\frac{1}{4}}$.

(13) Express $\frac{1}{29}$ of 1 lb. troy + $\frac{1}{29}$ of 1 lb. av. both as troy and as av. weights.

(14) Multiply $49\frac{15}{16}$ by $50\frac{1}{16}$, and add $\frac{1}{256}$ to the result.

(15) Reduce $\frac{3503}{4294}$ to lowest terms.

(16) Arrange in descending order of magnitude, $\frac{13}{25}$, $\frac{15}{26}$, $\frac{16}{27}$.

(17) If the step of a man be $2\frac{1}{3}$ feet, and that of a horse $2\frac{3}{4}$ feet, how many horse-paces are equal to 50 man-paces?

(18) Find the smallest exact number of feet which is an exact number of horse-paces and of man-paces.

(19) Find the largest quantity which is contained a whole number of times in each of the following: $2\frac{5}{9}$, $6\frac{7}{18}$. $11\frac{1}{2}$, $19\frac{1}{6}$.

(20) Find the value of £$\frac{1}{1000}$ + $\frac{1}{100}d$.

(21) The area of a certain room is $265\frac{5}{6}$ square feet; its length is $17\frac{3}{8}$ feet. Find its breadth.

(22) The volume of a log of wood is 115 cubic feet; its breadth is $3\frac{3}{8}$ feet; its thickness, $11\frac{1}{2}$ inches. Find the length.

(23) What fraction of 12*s.* 6*d.* must be added to $\frac{5}{7}$ of 1 guinea to make £1?

(24) Multiply the sum of $\frac{2}{3}$ of £100, and $\frac{1}{2\frac{2}{3}}$ of £6. 6*s.* 8*d.*, by $\frac{2}{7}$ of $\frac{\frac{5}{8}+\frac{2}{9}}{1\frac{1}{23}}$.

(25) Find L.C.M. of $\frac{3}{14}$, $1\frac{9}{35}$, $\frac{33}{56}$, $3\frac{20}{63}$.

(26) On Monday I spent $\frac{3}{8}$ of my money, on Tuesday $\frac{2}{5}$ of the original sum, and had then £11. 12*s.* 6*d.* left. How much had I at first?

(27) If on Tuesday I had spent $\frac{2}{3}$ of what was left me from Monday, and had then had £11. 12*s.* 6*d.* left, what would my original sum have been?

(28) Simplify $\frac{2\frac{2}{3}+1\frac{1}{4}}{2\frac{2}{3}-1\frac{1}{4}} \div \frac{5}{8}$.

(29) A person bequeathed $\frac{5}{12}$ of his property to A, $\frac{1}{4}$ of it to B, $\frac{1}{6}$ to C, and $\frac{1}{8}$ to D. The remainder, £550, to charities. What was the value of the whole property?

(30) If 1 bushel last $3\frac{2}{7}$ days, how many days will $4\frac{11}{20}$ bushels last?

(31) Divide the sum of $8\frac{4}{7}$ and $4\frac{4}{9}$ by

$\left.\begin{array}{l} a.\ \text{the sum} \\ b.\ \text{the difference} \\ c.\ \text{the product} \end{array}\right\}$ of their reciprocals.

(32) Machine A can pump 3 gallons in 5 minutes, machine B works half as fast again as A, and C at half B's speed. In what time would A alone pump 1 gallon?

B „ 4 gallons?

C „ $1\frac{1}{2}$ gallons?

A, B and C together $2\frac{5}{8}$ gallons?

(33) Divide the sum of $\frac{2}{3}$ of $3\frac{3}{10}$, $\frac{1\frac{3}{4}}{2\frac{5}{8}}$ of 17, and $\frac{3}{5}$ of $5\frac{3}{4}$ of $\frac{34}{51}$, by 19.

(34) Simplify $\frac{15}{17}$ of $\frac{6\frac{4}{5}}{2\frac{8}{11}}$ of $(2\frac{7}{9} \div 3\frac{8}{9})$.

(35) Express $\frac{1}{5}$ of 13*s.* 4*d.* as a fraction of £5.

(36) If $\frac{3}{7}$ of an estate be worth £450, what is the worth of $\frac{11}{24}$ of it?

[$\frac{3}{7}$ is worth £450;
1 „ $\frac{7}{3}$ of £450;
$\frac{11}{24}$ „ $\frac{11}{24} \times \frac{7}{3} \times$ £450.]

(37) A can do a piece of work in 10 days, B can do it in 12 days, and C in 9 days. In what time will all three do $2\frac{1}{2}$ such pieces of work? What share of all the work is done by A, B and C respectively? If £1. 10*s.* 11*d.* is paid per piece of work, how much should each receive for the $2\frac{1}{2}$ pieces?

(38) A man can do a piece of work in 5 days, which a woman would take 8 and a boy 12 days to finish; the man worked $1\frac{1}{4}$ days and was joined by the woman; both together then worked for $1\frac{1}{2}$ days, leaving the remainder to be finished by the boy. How long will he take to complete his task, and which of the three will have done the largest and which the smallest share of the work?

(39) Give three different interpretations to the symbol $\frac{3}{4}$, and apply each to $\frac{4\frac{1}{5}}{6\frac{1}{8}}$.

(40) I had $\frac{2}{3}$ of a ship and sold $\frac{4}{5}$ of this share for £1200. What is the value of the whole ship?

(41) If 220 gals. of creosote, at 1*d.* per gal., give as much heat as $2\frac{1}{2}$ tons of coal, what will be the cost of the quantity of creosote that has the heating power of 1 ton of coal?

(42) The value of an oz. of standard gold is £3. 17*s.* $10\frac{1}{2}$*d.* What fraction of £1000 are 625 oz. of gold?

(43) Find the value of $\frac{2}{3}$ of $\frac{5}{8}$ of $\frac{1}{2\frac{1}{3}}$ of 7 articles, if $\frac{3}{5}$ of $1\frac{3}{4} \times 2\frac{1}{3}$ articles cost £15. 7*s.* 8*d.*?

(44) A cistern of 960 gallons is emptied by two pipes, A and B, in 5 and 7 minutes respectively. How much water will pass through each pipe if both are opened together?

(45) A and B together can do a piece of work in 6 days, A and

C in 8 days, B and C in 9 days. In what time could all three together do it?

(46) A brick is 9 in. long, $4\frac{1}{2}$ in. wide, and 3 in. thick. How many bricks are wanted to build a wall 520 yds., 9 in. long, 15 ft. high and $1\frac{1}{2}$ ft. thick?

(47) What will be the cost of painting the four walls of a room which is 24 ft., 3 in. long, 11 ft., 9 in. broad, and 11 ft., 6 in. high, at 1*s.* 6*d.* per sq. ft.?

(48) Find the average of $21\frac{2}{3}$, $73\frac{4}{5}$, 0, $3\frac{13}{200}$, 82, $17\frac{3}{20}$, $5\frac{1}{4}$, $9\frac{5}{12}$.

(49) If I spend on the first day $\frac{3}{5}$ of my money, next day $\frac{3}{5}$ of what is left, and so on for 4 days, what fraction of the original sum will be left?

(50) By what must the difference between $\frac{51}{88}$ and $\frac{13}{55}$ be

$$\left.\begin{array}{l} a.\ \text{increased,} \\ b.\ \text{multiplied,} \\ c.\ \text{divided,} \end{array}\right\}\ \text{to give 12?}$$

(51) Divide 15*s.* 9*d.* by $2\frac{5}{8}$.

(52) Divide 15*s.* 9*d.* by 2*s.* $7\frac{1}{2}$*d.*

(53) Divide 15*s.* 9*d.* by £$2\frac{5}{8}$.

(54) A cistern can be filled by pipes A and B in 5 and 6 minutes respectively, and emptied by C in 4 minutes. In what time will the cistern be filled if all three are opened?

(55) From 1 lb. troy are coined $46\frac{29}{40}$ sovs. Express the weight of 1 sov., both by troy and av. weight.

(56) Simplify $$\frac{1\frac{13}{20} \times 4 + \dfrac{2\frac{1}{40}}{1\frac{4}{5}} + \left(\frac{3}{20} \div \frac{3}{22}\right)}{\dfrac{(4\frac{1}{5} \times 6\frac{2}{3}) + (11\frac{1}{5} \times 1\frac{2}{3})}{11\frac{1}{5} \times 6\frac{2}{3}}} \times \frac{2\frac{1}{3}}{1\frac{5}{8}} \text{ of } \frac{1\frac{5}{8}}{2\frac{1}{3}}$$

(57) Simplify $$\frac{1}{6 + \frac{1}{7\frac{1}{3}}} + \frac{3}{5} \text{ of } 1\frac{2}{5} + \left(1\frac{2}{7} \div 1\frac{1}{4}\right) + \frac{9}{350}.$$

(58) The area of a certain floor is $145\frac{1}{2}$ sq. ft.; its length is 15 ft., $4\frac{1}{2}$ in. Find the width.

(59) A court-yard is to be paved with tiles $10\frac{1}{2}$ in. square. How many tiles will be wanted if the court is 7 yds., 2 ft., 4 in. long, and 4 yds., 2 feet wide?

(60) How many tiles would have been wanted if each had an area of $10\frac{1}{2}$ sq. in.?

(61) State and prove the rules for the multiplication and division of one vulgar fraction by another. Shew that the multiplication of two proper fractions will give a product less than either of them.

(62) Simplify

$\alpha.\ \dfrac{m\times a}{b\times m}.$ $\qquad\zeta.\ \dfrac{m\times a+m\times b}{m}.$

$\beta.\ \dfrac{a}{b\times c}+\dfrac{b}{c\times a}+\dfrac{c}{b\times a}.$ $\qquad\eta.\ \dfrac{1}{a+\frac{1}{b}}.$

$\gamma.\ \dfrac{a}{b}\times\dfrac{c}{d}.$ $\qquad\theta.\ \left(\dfrac{a}{b}+\dfrac{c}{d}\right)\times b.$

$\delta.\ \dfrac{a}{b}\div\dfrac{c}{d}.$ $\qquad\iota.\ \left(\dfrac{a}{b}-\dfrac{c}{d}\right)\times d.$

$\epsilon.\ \dfrac{a}{b}\times\dfrac{c}{d}\times b\times d.$ $\qquad\kappa.\ \left(\dfrac{a}{b}\pm\dfrac{c}{d}\right)\times b\times d.$

(63) Simplify $\dfrac{\frac{27}{64}-\frac{8}{27}}{\frac{9}{16}+\frac{1}{2}+\frac{4}{9}}+\dfrac{4\frac{1}{7}-2\frac{1}{4}}{6\frac{1}{2}-2\frac{1}{7}}.$

(64) Divide 1 lb. troy and also 1 lb. avoirdupois by 17 lb. 10 oz., 6 dwts., 15 grs. troy.

(65) A room is 8 yds., 2 ft., 3 in. long, and 5 yds., 9 in. broad. Find the cost of covering it with carpet $\frac{3}{4}$ yd. wide, at 4*s.* 6*d.* a yd.

(66) If the value of one rupee is $1\frac{23}{24}$*s.* how many rupees can be bought for £$7\frac{14}{15}$?

(67) If 56 cubic feet, 1044 cubic inches, of timber are required to floor a room $29\frac{1}{4}$ feet long, and $25\frac{1}{3}$ feet wide, what is the thickness of the boards?

(68) Find the fraction of £1 which is equivalent to the excess of $\frac{2}{3}$ of a guinea above the sum of $\frac{3}{4}$ of 1*s.* and $\frac{4}{9}$ of 7*s.* 6*d.*

(69) The cargo of a ship worth £45000 belongs to three partners; A owns $\frac{7}{9}$ of $\frac{3}{5}$ of it; B's share is equal to $3\frac{3}{14}$ times $\frac{2}{9}$ of A's share, and C owns the remainder. What part of the cargo is owned by each partner, and what ought each to receive from the sale?

(70) If the circumference of a wheel is $3\frac{1}{7}$ times its diameter, how many times will a wheel 1 yd., $1\frac{1}{8}$ ft. in diameter, revolve in travelling $3\frac{3}{7}$ miles?

CHAPTER V.

PRACTICE.

§ 1. Practice is another mode of performing multiplication.

CASE I. Find the cost of 428 articles at 2*s.* each.

2*s.* $= \frac{1}{10}$ of £1, ∴ 428 × 2*s.* = £$\frac{428}{10}$ = £42$\frac{8}{10}$ = £42. 8 florins = £42. 16*s.* Similarly 3259 × 2*s.* = £325. 9 florins = £325. 18*s.*, and any number of times 2*s.* may be obtained by doubling the last figure for the shillings, and calling the rest pounds.

EXERCISE XXXV.

(1) 371 × 2*s.*	(6) 4572 × 2*s.*
(2) 8643 × 2*s.*	(7) 9724 × 2*s.*
(3) 615 × 2*s.*	(8) 1376 × 2*s.*
(4) 8497 × 2*s.*	(9) 7948 × 2*s.*
(5) 1729 × 2*s.*	(10) 320 × 2*s.*

CASE II. Find the cost of 367 articles at 12*s.* each.

12*s.* = 6 florins, 367 × 12*s.* = (367 × 1 fl.) × 6 = (£36. 7 fl.) × 6 = £216. 42 fl. = £220. 2 fl. = £220. 4*s.*

Wording: 36,7 × 6 = 42 (florins) **4′**, carry 4; 36, **40′**, carry 4; 18, **22′**.

5493 × 14*s.* 549,3 × 7 = £3845. 2*s.*

Wording: 21, **2′**, carry 2; 63, **65′**, carry 6; 28, **34′**, carry 3; 35, **38′**.

EXERCISE XXXVI.

(1) 4378 × 12*s.*	(6) 7341 × 4*s.*
(2) 987 × 14*s.*	(7) 8267 × 6*s.*
(3) 6716 × 16*s.*	(8) 9752 × 8*s.*
(4) 3545 × 18*s.*	(9) 723 × 12*s.*
(5) 3714 × 8*s.*	(10) 8769 × 10*s.*

CASE III. Find the cost of 1389 articles at 1*s.* each.

1389 × 1*s.* = $\frac{1}{2}$ of 1389 × 2*s.* = $\frac{1}{2}$ of £138. 9 fl. = £69. 9*s.*

Wording: 2 in 13, **6′**, carry 1; in 18, **9′**; in 9 fl., **9′*s.***

7356 × 1*s.* = $\frac{1}{2}$ of £735. 6 fl. = £367. 16*s.*

Wording: 2 in 7, **3′**, carry 1; in 13, **6′**, carry 1; in 15, **7′**, carry 1; in 16 fl., **16′*s.***

Exercise XXXVII.

(1) $9802 \times 1s.$
(2) $8491 \times 1s.$
(3) $4624 \times 1s.$
(4) $3765 \times 1s.$
(5) $537 \times 1s.$
(6) $1280 \times 1s.$
(7) $1579 \times 1s.$
(8) $658 \times 1s.$
(9) $713 \times 1s.$
(10) $5046 \times 1s.$

Case IV. Find the cost of 4769 articles at 17*s.* each.

$4769 \times 16s. =$	£3815	4	0
,, $\times 1s. =$	238	9	0
,, $\times 17s. =$	£4053	13	0

Exercise XXXVIII.

(1) $4253 \times 3s.$
(2) $8674 \times 7s.$
(3) $2587 \times 9s.$
(4) $483 \times 13s.$
(5) $5724 \times 17s.$
(6) $3048 \times 19s.$
(7) $956 \times 11s.$
(8) $5010 \times 17s.$
(9) $3466 \times 19s.$
(10) $8888 \times 3s.$

Case V. Aliquot Parts. An aliquot part of any quantity is a measure of that quantity. Thus 6*s.* 8*d.*, being $\frac{1}{3}$ of £1, is an aliquot part of £1. Similarly $1\frac{1}{2}d.$ ($\frac{1}{4}$ of 6*d.*) is an aliquot part of 6*d.* 21 lbs. ($\frac{1}{16}$ of 3 cwts.) is an aliquot part of 3 cwts. The useful aliquot parts of £1 are 10*s.*, 6*s.* 8*d.*, 5*s.*, 4*s.*, 3*s.* 4*d.*, 2*s.* 6*d.*, 2*s.*, 1*s.* 8*d.*, 1*s.*

Find the cost of 419 articles at 3*s.* 4*d.* each (3*s.* 4*d.* $= \frac{1}{6}$ of £1.)
$419 \times 3s.\ 4d. = £\frac{419}{6} = £69.\ 16s.\ 8d.$

Exercise XXXIX.

(1) $846 \times 10s.$
(2) $789 \times 4s.$
(3) $8736 \times 2s.\ 6d.$
(4) $1234 \times 1s.\ 8d.$
(5) $477 \times 3s.\ 4d.$
(6) $3124 \times 5s.$
(7) $385 \times 6s.\ 8d.$
(8) $4457 \times 2s.\ 6d.$
(9) $1494 \times 3s.\ 4d.$
(10) $779 \times 6s.\ 8d.$
(11) $387 \times 10s.$
(12) $7773 \times 2s.\ 6d.$
(13) $765 \times 1s.\ 8d.$
(14) $1452 \times 2s.\ 6d.$
(15) $17 \times 10s.$
(16) $2831 \times 5s.$
(17) $43 \times 6s.\ 8d.$
(18) $787 \times 3s.\ 4d.$
(19) $29 \times 5s.$
(20) $76047 \times 3s.\ 4d.$
(21) $451 \times 2s.\ 6d.$
(22) $4284 \times 6s.\ 8d.$
(23) $2318 \times 5s.$
(24) $51 \times 4s.$
(25) $1717 \times 5s.$
(26) $11 \times 3s.\ 4d.$

(27) 1837 × 4*s.*
(28) 2655 × 3*s.* 4*d.*
(29) 2470 × 2*s.* 6*d.*
(30) 4776 × 1*s.* 8*d.*
(31) 38 × 2*s.* 6*d.*
(32) 588 × 1*s.* 8*d.*
(33) 848 × 3*s.* 4*d.*
(34) 3786 × 2*s.* 6*d.*
(35) 567 × 2*s.* 6*d.*
(36) 11111 × 1*s.* 8*d.*

CASE VI. The useful aliquot parts of 2*s.* are 1*s.*, 8*d.*, 6*d.*, 4*d.*, 3*d.*, 2*d.*; those of 1*s.* are 6*d.*, 4*d.*, 3*d.*, 2*d.*, $1\frac{1}{2}d.$ and 1*d.*; of 6*d.* are 3*d.*, 2*d.*, $1\frac{1}{2}d.$, 1*d.*, $\frac{3}{4}d.$, $\frac{1}{2}d.$

Find the cost of 477 articles at 8*d.* each.

477 × 8*d.* = $\frac{1}{3}$ of £47. 7 fl. = £15. 18*s.*

Wording: 3 in 4, **1′**, carry 1; in 17, **5′**, carry 2; in 27 fl., 9 fl. = **18′***s.*

Find the cost of 643 articles at 8*d.* each.

64‚3 × 8*d.* = £21. 8*s.* 8*d.*

Wording: 3 in 6, **2′**; in 4, **1′**, carry 1; (13 fl. = 26*s.*); in 26, **8′**, carry 2*s.*; in 24, **8′**.

Find the cost of 5261 articles at 8*d.* each.

526‚1 × 8*d.* = £175. 7*s.* 4*d.*

N.B. Compare the wording of the first with that of the two following examples here given. In the first the number of florins is divisible by 3.

EXERCISE XL.

(1) 5430 × 8*d.*
(2) 5124 × 2*d.*
(3) 1524 × 6*d.*
(4) 1740 × 6*d.*
(5) 3432 × 4*d.*
(6) 41296 × 8*d.*
(7) 3456 × 8*d.*
(8) 26413 × 2*d.*
(9) 5899 × 6*d.*
(10) 4106 × 3*d.*
(11) 2745 × 4*d.*
(12) 12113 × 3*d.*
(13) 5468 × 2*d.*
(14) 6794 × 8*d.*
(15) 8142 × 6*d.*
(16) 8182 × 2*d.*
(17) 4533 × 6*d.*
(18) 2330 × 4*d.*
(19) 6658 × 8*d.*
(20) 4507 × 4*d.*
(21) 8155 × 3*d.*
(22) 7346 × 6*d.*
(23) 3751 × 2*d.*
(24) 4414 × 4*d.*
(25) 9255 × 3*d.*
(26) 7646 × 8*d.*
(27) 5273 × 2*d.*
(28) 2851 × 8*d.*
(29) 7108 × 3*d.*
(30) 6498 × 2*d.*
(31) 2537 × 4*d.*
(32) 9059 × 6*d.*
(33) 1439 × 2*d.*
(34) 3413 × 6*d.*
(35) 17222 × 3*d.*
(36) 21329 × 3*d.*

CASE VII. Find the cost of 1749 articles at £3. 7*s*. 9¾*d*. each.

A	1749 × £3. =	£5247	0	0
B	,, × 6*s*. =	524	14	0
C	,, × 1*s*. =	87	9	0
D	,, × 6*d*. =	43	14	6
E	,, × 3*d*. =	21	17	3
F	,, × ¾*d*. =	5	9	3¾
		£5930	4	0¾

D is obtained by halving C,
E " " D,
F " finding the quarter of E.

Find the cost of 2359 articles at £1. 14*s*. 7¾*d*. each.

A	2359 × £1. =	£2359	0	0
B	,, × 14*s*. =	1651	6	0
C	,, × 6*d*. =	58	19	6
D	,, × 1½*d*. =	14	14	10½
E	,, × ¼*d*. =	2	9	1¾
		£4086	9	6¼

C is found as in Case VI.
D " by dividing C by 4,
E " " D by 6.

Find the cost of 947 articles at 16*s*. 11¾*d*. each.

A	947 × 16*s*. =	£757	12	0
B	,, × 8*d*. =	31	11	4
C	,, × 3*d*. =	11	16	9
D	,, × ¾*d*. =	2	19	2¼
		£803	19	3¼

B and C are obtained as in Case VI.
D is one-fourth of C.

EXERCISE XLI.

			£.	s.	d.				£.	s.	d.
(1)	7000	@	5	10	0	(7)	529	@	4	8	7½
(2)	394		1	18	0	(8)	7641		3	7	11¾
(3)	1776		4	17	6	(9)	82465		0	13	9¼
(4)	3458		2	13	4	(10)	8762		10	1	1½
(5)	43728		8	18	6	(11)	385		5	7	6¾
(6)	4639		1	17	3	(12)	1044		3	19	10½

			£.	s.	d.				£.	s.	d.
(13)	157	@	6	18	$9\frac{3}{4}$	(20)	854	@	0	5	3
(14)	7235		10	16	$7\frac{1}{4}$	(21)	720		11	10	$7\frac{1}{4}$
(15)	586		4	4	$4\frac{1}{2}$	(22)	37		10	6	$8\frac{1}{2}$
(16)	725		0	12	$11\frac{1}{2}$	(23)	256		20	11	$4\frac{3}{4}$
(17)	678		0	9	$2\frac{1}{4}$	(24)	749		40	14	6
(18)	965		3	2	$5\frac{1}{2}$	(25)	365		0	15	$5\frac{1}{4}$
(19)	132		7	3	$8\frac{3}{4}$	(26)	1870		8	14	$9\frac{1}{2}$

(27) Taking a year to be 365 days, find the wages for a year, if the daily wages be 1*s*., 4*s*. 6*d*., 7*s*. 4*d*., £1. 2*s*. 6*d*., £3. 4*s*. 10*d*.

CASE VIII. Find the cost of $2037\frac{1}{2}$ articles at £4. 11*s*. 7*d*. each.

2037 × £4. =	£8148	0	0	
,, × 10*s*. =	1018	10	0	
,, × 1*s*. =	101	17	0	
,, × 6*d*. =	50	18	6	
,, × 1*d*. =	8	9	9	
$\frac{1}{2}$ of £4. 11*s*. 7*d*. =	2	5	$9\frac{1}{2}$	
	£9330	1	$0\frac{1}{2}$	

$2675\frac{4}{9}$ articles at £1. 18*s*. $3\frac{1}{2}$*d*.

2675 × £1. =	£2675	0	0	
,, × 18*s*. =	2407	10	0	
,, × 3*d*. =	33	8	9	18
,, × $\frac{1}{2}$*d*. =	5	11	$5\frac{1}{2}$	9
$\frac{1}{9}$ of £1. 18*s*. $3\frac{1}{2}$*d*. =	0	4	$3\frac{1}{18}$	1
$\frac{3}{9}$,, =	0	12	$9\frac{1}{6}$	3
	£5122	7	$2\frac{13}{18}$	13

Similarly, if the fraction of the article were $\frac{4}{5}$, $\frac{8}{13}$, &c., we may first find the cost of $\frac{1}{5}$, $\frac{1}{13}$, &c., and then by multiplication the cost of the remaining $\frac{3}{5}$, $\frac{7}{13}$, &c.

EXERCISE XLII.

(1) $426\frac{1}{2}$ × £8. 4*s*. $3\frac{1}{2}$*d*.

(2) $712\frac{2}{3}$ × £5. 7*s*. 4*d*.

(3) $28\frac{15}{16}$ × £20. 4*s*. 4*d*.

(4) $537\frac{7}{11}$ × £33. 4*s*. 9*d*.

(5) $821\frac{5}{8}$ × £7. 10*s*. $3\frac{3}{4}$*d*.

(6) $246\frac{7}{20}$ × £14. 4*s*. $4\frac{1}{4}$*d*.

CASE IX.* Find the cost of 9 cwt., 3 qrs., 18 lbs. at £7. 10*s.* 8*d.* per cwt.

A	9 cwt. at £7. 10s. 8d.	=	£67	16	0
B	2 qrs. ,,	=	3	15	4
C	1 qr. ,,	=	1	17	8
D	14 lbs. ,,	=	0	18	10
E	4 lbs. ,,	=	0	5	$4\frac{4}{7}$
			£74	13	$2\frac{4}{7}$

B. 2 qrs. = $\frac{1}{2}$ of 1 cwt. Find $\frac{1}{2}$ of £7. 10*s.* 8*d.*
C. 1 qr. = $\frac{1}{2}$ of 2 qrs. ,, $\frac{1}{2}$ of B.
D. 14 lbs. = $\frac{1}{2}$ of 1 qr. ,, $\frac{1}{2}$ of C.
E. 4 lbs. = $\frac{1}{7}$ of 1 qr. ,, $\frac{1}{7}$ of C.

Find the value of 7 oz., 14 dwts., 11 grs. of gold, at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.

							320
7 oz. at £3. 17s. $10\frac{1}{2}$d.	=	£27	5	$1\frac{1}{2}$		—	160
10 dwts. ,,	=	1	18	$11\frac{1}{4}$		—	80
4 ,, ,,	=	0	15	$6\frac{9}{16}$		32	288
8 grs. ,,	=	0	1	$3\frac{23}{40}$		8	184
2 ,, ,,	=	0	0	$3\frac{143}{160}$		2	286
1 gr. ,,	=	0	0	$1\frac{303}{320}$		—	303
		£30	1	$5\frac{21}{320}$			$\frac{1301}{320}=4\frac{21}{320}$.

EXERCISE XLIII.

(1) 7 tons, 5 cwt., 1 qr., 21 lbs. at £10. 4*s.* per ton.
(2) 11 cwt., 3 qrs., 14 lbs. at £8. 10*s.* per ton.
(3) 4 cwt., 1 qr., 17 lbs. at £3. 3*s.* per cwt.
(4) 4 oz., 11 dwt., 17 grs. at £5. 10*s.* per oz.
(5) 14 miles, 2 fur., 143 yds. at £42 per mile.
(6) 3 qrs., 19 lbs. at £7. 8*s.* 4*d.* per cwt.
(7) 4 hours, 17 min., 17 sec. at 45 miles per hour.
(8) 11 tons, 11 cwt., 23 lbs. at £3. 19*s.* per ton.
(9) 4 oz., 7 dwts., 11 grs. at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.
(10) 23 lbs., 14 oz. av. at £18. 10*s.* per cwt.

* Students who intend to pursue the study of Arithmetic into Part III. are advised not to waste much time over this barbarous rule, which we give reluctantly in deference to use and wont.

CASE X. Find the cost of 149 tons, 13 cwt., 3 qrs., 10 lbs., at £43. 8*s.* 4*d.* per ton.

		£	s.	d.		112
149 × £43	=	£6407	0	0		
,, × 8*s.*	=	59	12	0		
,, × 4*d.*	=	2	9	8		
10 cwt. at £43. 8*s.* 4*d.* per ton		21	14	2		
2 ,, ,,	=	4	6	10		
1 ,, ,,	=	2	3	5		
2 qrs. ,,	=	1	1	$8\frac{1}{2}$	—	56
1 qr. ,,	=	0	10	$10\frac{1}{4}$	—	28
7 lbs. ,,	=	0	2	$8\frac{9}{16}$	7	63
1 lb. ,,	=	0	0	$4\frac{73}{112}$	—	73
2 lbs. ,,	=	0	0	$9\frac{17}{56}$	2	34
		£6499	2	$6\frac{15}{56}$		$\frac{254}{112}=2\frac{30}{112}\vert\frac{15}{56}$.

Find the dividend on £537. 8*s.* 10*d.* at 9*s.* $7\frac{3}{4}$*d.* in the £.

		£	s.	d.		480
Dividend of 20*s.* in the £	=	£537	8	10		
,, 5*s.* ,,	=	134	7	$2\frac{1}{2}$	—	240
,, 4*s.* ,,	=	107	9	$9\frac{1}{5}$	—	96
,, 6*d.* ,,	=	13	8	$8\frac{13}{20}$	24	312
,, $1\frac{1}{2}$*d.* ,,	=	3	7	$2\frac{13}{80}$	6	78
,, $\frac{1}{4}$*d.* ,,	=	0	11	$2\frac{173}{480}$	—	173
		£259	4	$0\frac{419}{480}$		$\frac{899}{480}=1\frac{419}{480}$*d.*

EXERCISE XLIV.

(1) 421 tons, 11 cwt. at £5. 12*s.* 6*d.* per ton.
(2) 75 tons, 9 cwt., 1 qr. at £12. 10*s.* 6*d.* per ton.
(3) 137 oz., 12 dwt. at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.
(4) 511 miles, 5 fur., 77 yds. at £417. 18*s.* 9*d.* per mile.
(5) 4*s.* $10\frac{1}{2}$*d.* in the £ on £5811. 17*s.* 6*d.*
(6) 71 days (of 12 hrs.), $9\frac{1}{2}$ hrs. at £7. 7*s.* a day.
(7) 111 yds., 1 ft., 1 in. at 5*s.* $6\frac{1}{2}$*d.* per yd.
(8) 5*s.* $4\frac{3}{4}$*d.* in the £ on £826. 14*s.* 10*d.*
(9) 37 cwt., 3 qrs., 24 lbs. at £10. 12*s.* 8*d.* per cwt.
(10) 76 gals., 3 qts., 1 pt. at 14*s.* 9*d.* per gal.
(11) 23 years, 17 weeks, at £10. 10*s.* a year.
(12) Find the price of 14 ingots of gold, each weighing 3 lbs., 7 oz., 14 dwts., 21 grs. at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.

(13) 3 cwt., 2 qrs., 16 lbs. at £3. 7*s.* 8*d.* per cwt.

(14) 3 acres, 1 rood, 14 poles, at £125 per acre.

(15) 15 silver plates, each weighing 7 oz., 11 dwts., 6 grs. at 6*s.* 8*d.* per oz.

CHAPTER VI.

PROPORTION.

§ 1. The Unitary Method.

Find the cost of 27 articles, if 12 cost £4. 17*s.* 6*d.*

Statement : 12 articles cost £4. 17*s.* 6*d.*
Question : 27 „ £x,

where x stands for "the quantity to be found."

Mod. op.: 12 articles cost............ £4. 17*s.* 6*d.*
1 article costs ...$\frac{1}{12}$ of £4. 17*s.* 6*d.*
27 articles cost ...$\frac{27}{12}$ of £4. 17*s.* 6*d.*

$$\frac{\overset{9}{\cancel{27}}}{\underset{4}{\cancel{12}}} \text{ of } £4.\ 17s.\ 6d. = \frac{£43.\ 17s.\ 6d.}{4} = £10.\ 19s.\ 4\tfrac{1}{2}d.$$

Ans. £10. 19*s.* 4½*d.*

How many articles can be bought for £28, if 11 articles cost £15. 8*s.*?

Statement : £15. 8*s.* buy 11 articles.
Question : £28 „ x „

Mod. op.: £15. 8*s.* = 308*s.* buy ....11* articles.
1*s.* buys......$\frac{1}{308}$ of 11 „
£28. 0*s.* = 560*s.* buy ......$\frac{560}{308}$ of 11 „

$$\frac{\overset{\overset{20}{\cancel{140}}}{\cancel{560}} \times \overset{1}{\cancel{11}}}{\underset{\underset{\underset{1}{\cancel{11}}}{\cancel{77}}}{\cancel{308}}} = 20.$$

Ans. 20 articles.

* Always place last the quantity of the same denomination as the answer required.

If 3 cwt., 3 qrs., 21 lbs. cost £3. 18*s*. 9*d*., what will 2 tons, 17 cwt., 3 qrs., 10 lbs. cost?

Statement: 3 cwt., 3 qrs., 21 lbs. cost............£3. 18*s*. 9*d*.
Question: 2 tons, 17 cwt., 3 qrs., 10 lbs. cost £*x*.

Mod. op.:

3 cwt., 3 qrs. 21 lbs. = 441 lbs. cost...............£3. 18*s*. 9*d*.
1 lb. costs... $\frac{1}{441}$ of £3. 18*s*. 9*d*.
2 tons, 17 cwt., 3 qrs., 10 lbs. = 6478 lbs. cost ... $\frac{6478}{441}$ of £3. 18*s*. 9*d*.

$\frac{6478}{441}$ of £3. 18*s*. 9*d*. = £57. 16*s*. $9\frac{3}{7}$*d*.

Ans. £57. 16*s*. $9\frac{3}{7}$*d*.

If 17 men require $4\frac{2}{5}$ days to build a certain wall, how long will 13 men take?

Statement: 17 men require $4\frac{2}{5}$ days.
Question: 13 " *x* "

Mod. op.:
17 men require ....$4\frac{2}{5}$ days.
1 man requires............$17 \times 4\frac{2}{5}$ "
13 men require......$\frac{1}{13}$ of $17 \times 4\frac{2}{5}$ "

$$\frac{17 \times 4\frac{2}{5}}{13} = \frac{374}{65} = 5\frac{49}{65}.$$

Ans. $5\frac{49}{65}$ days.

If of carpet $\frac{3}{4}$ yd. wide, I require $62\frac{1}{2}$ yds. to cover a room, how much shall I require of carpet $1\frac{1}{8}$ yds. wide?

Statement: With a width of $\frac{3}{4}$ yd. I require $62\frac{1}{2}$ yds.
Question: " $1\frac{1}{8}$ " *x* "

Mod. op.:
With $\frac{3}{4} = \frac{6}{8}$ width, I require............$62\frac{1}{2}$ yds.
" $\frac{1}{8}$ " $6 \times 62\frac{1}{2}$ "
" $1\frac{1}{8} = \frac{9}{8}$ " $\frac{1}{9}$ of $6 \times 62\frac{1}{2}$ "

$$\frac{\overset{2}{\cancel{6}} \times 62\frac{1}{2}}{\underset{3}{\cancel{9}}} = \frac{125}{3} = 41\frac{2}{3}.$$

Ans. $41\frac{2}{3}$ yds.

Exercises XLVIII., XLIX., L. and LI., will supply examples for practising this method.

Examination of the work here given will shew that we always have to take a certain fraction of the quantity which is of like denomination with the answer, and what this fraction is can be more concisely and elegantly found by "Proportion," to which we now proceed.

§ 2. PROPORTION.

Find a number that shall be the same fraction of 12 that 6 is of 8. 6 is $\frac{6}{8}$ or $\frac{3}{4}$ of 8, ∴ we have to find $\frac{3}{4}$ of 12. *Ans.* 9.

EXERCISE XLV.

What quantity is the same fraction

(1)	of 20	that 4	is of 5?
(2)	„ 35	„ 6	„ 15?
(3)	„ 35	„ 15	„ 6?
(4)	„ $2\frac{3}{4}$	„ $1\frac{1}{2}$	„ $1\frac{3}{4}$?
(5)	„ $10\frac{4}{5}$	„ $1\frac{4}{5}$	„ 2?
(6)	„ $10\frac{4}{5}$	„ $10\frac{4}{5}$	„ 11?
(7)	„ £2. 10*s.* 8*d.*	„ 3	„ 4?
(8)	„ £10. 10*s.*	„ 1 cwt., 3 qrs.	„ 2 cwt., 2 qrs.?
(9)	„ £8. 7*s.* 6*d.*	„ 2 yrs., 73 d.	„ $3\frac{1}{2}$ yrs.?
(10)	„ 1 oz. troy	„ 1*s.*	„ £1?
(11)	„ 1 oz. troy	„ 1*d.*	„ £1?
(12)	„ 1 lb. troy	„ 1*s.*	„ £1?
(13)	„ £1	„ 1 oz.	„ 1 lb. troy?
(14)	„ £1	„ 1 dwt.	„ 1 lb. troy?
(15)	„ £1	„ 1 qr.	„ 1 ton?
(16)	„ £1	„ 1 fur.	„ 1 mile?
(17)	„ 2*s.* 6*d.*	„ 1 pole	„ 1 furlong?
(18)	„ £1. 1*s.*	„ $6\frac{2}{3}$ yds.	„ $3\frac{4}{5}$ furlongs?
(19)	„ $\frac{1}{4\frac{2}{7}}$	„ $3 \times 1\frac{2}{7}$	„ $6 \times 1\frac{2}{7}$?
(20)	„ The reciprocal of 10	„ 11	„ its reciprocal?
(21)	„ 1000 sheep	„ 3 men	„ 5 men?

§ 3. The fraction that one number is of another is called the RATIO of the first number to the second. In every ratio, therefore, there must be two quantities (or "terms") of which the first is called the *Antecedent* and the second is called the *Consequent.*

Of the ratio $\frac{3}{4}$, the numerator 3 is the antecedent, 4 the consequent. The ratio $\frac{3}{4}$ is commonly written 3 : 4, and is read 3 *to* 4.

We see that 3 : 4 is one more interpretation of $\frac{3}{4}$, and we have the following sets of interchangeable terms. (Cf. p. 21.)

Numerator,	Denominator,	Fraction,
Dividend,	Divisor,	Quotient,
Antecedent,	Consequent,	Ratio.

Note, however, that every ratio must be an abstract number, and hence that the Antecedent and Consequent must be quantities of the same kind. Ratio, in fact, is Relative Magnitude, which can of course only subsist between quantities whose magnitudes can be compared.

§ 4. When two ratios are equal, the four terms are said to be IN PROPORTION; thus if $a : b = c : d$, a, b, c, d are proportional. In every proportion there must therefore be four terms. Of these, the 1st and 4th are called ***Extremes***, the 2nd and 3rd ***Means***. It is also convenient to call the two extremes or the two means ***similar*** terms, and one extreme and one mean ***dissimilar*** terms. Thus if $a : b = c : d$, a and d are both extremes and therefore similar, b and c are both means and therefore similar; again, a and b, a and c, b and d, c and d are dissimilar. On examining § 2, we see that the question, "Find a number that shall be the same fraction of 12 that 6 is of 8," is equivalent to finding the first term of a proportion, the other three being known; and using the letter x to express the quantity to be found, the question may be written:

"Find x, if $x : 12 = 6 : 8$." *Ans.* 9.

Similarly if $x : 5 = 3 : 8$, x must be $\frac{3}{8}$ of $5 = \frac{3 \times 5}{8}$. Hence generally, the product of the second and third terms, divided by the fourth, yields the first, or if $a : b = c : d$, $a = \frac{b \times c}{d}$. Since $b \times c$ has to be divided by d to give a, $b \times c = d \times a$; $\therefore$ if $a : b = c : d$, $a \times d = b \times c$.

This equality is expressed in words thus: "*The product of the extremes equals the product of the means.*"

If any number be broken up into two pairs of factors (as $36 = 4 \times 9 = 3 \times 12$), these two pairs are the similar terms of a proportion ($4 : 3 = 12 : 9$), for if another number be substituted for any one of these terms, the product of the extremes will no longer equal the product of the means.

If, then, the product of the extremes equal the product of the means, the four numbers are proportional, and if not, not.

§ 5. The proportion $4 : 3 = 12 : 9$, can be stated in the following eight different ways, for in each of them the product of the extremes equals the product of the means.

$$
\begin{array}{rcl}
4 : 3 &=& 12 : 9 \\
4 : 12 &=& 3 : 9 \\
3 : 4 &=& 9 : 12 \\
3 : 9 &=& 4 : 12 \\
12 : 9 &=& 4 : 3 \\
12 : 4 &=& 9 : 3 \\
9 : 12 &=& 3 : 4 \\
9 : 3 &=& 12 : 4
\end{array}
$$

There are sixteen other ways of arranging the four figures, but in no one of them does this test hold.

Exercise XLVI.

(1) Break up 60 into two pairs of factors, and write down the resulting proportion in eight different ways.

(2) Do the same with 15, 21, 25, 100, $2\frac{5}{8}$.

§ 6. If any three terms of a proportion be given, the fourth can be found.

If the missing term be an extreme, the question is, What number multiplied by the other extreme gives the product of the means? *Ans.* The product of the means divided by the other extreme.

If the missing term be a mean, the question is, What number multiplied by the other mean gives the product of the extremes? *Ans.* The product of the extremes divided by the other mean. Or, in other words, The product of any two similar terms divided by one dissimilar term gives the other dissimilar term.

Thus if $a : b = c : d$, $a = \frac{b \times c}{d}$; $b = \frac{a \times d}{c}$; $c = \frac{a \times d}{b}$; $d = \frac{b \times c}{a}$.

§ 7. Since "the product of any two similar terms divided by one dissimilar term gives the other dissimilar term," the two similar terms to be multiplied will give the numerator of a fraction, and the given dissimilar term the denominator. Hence the dissimilar term

may be cancelled against *either* of the two similar terms, or if convenient the dissimilar term and *either* of the similar terms may be multiplied by the same number.

Thus if $8 : 10 = 6 : x$, then $4 : 5 = 6 : x$, and $2 : 5 = 3 : x$;
also $3 \times 8 : 3 \times 10 = 6 : x$, and $5 \times 8 : 10 = 5 \times 6 : x$,
and $3 \times 5 \times 8 : 3 \times 10 = 5 \times 6 : x$, &c.

Generally: If $a : b = c : d$, $m \times a : m \times b = c : d$, and $m \times a : b = m \times c : d$, &c., where m represents any multiplier, integral or fractional. In other words, "Any two dissimilar terms may be multiplied or divided by the same number without disturbing the proportion."

Exercise XLVII.

I. State the following proportion in the eight different ways and find the value of x from each way. $x : 10 = 12 : 15$.

II. Find the value of x from the following:

(1) $x : 2 = 3 : 4$.

(2) $2 : x = 3 : 4$.

(3) $2\frac{1}{2} : 3\frac{1}{2} = x : 4\frac{1}{2}$.

(4) $5\frac{1}{4} : 1 = 3 : x$.

(5) $4 : 5 = 5 : x$.

(6) $\dfrac{1}{2\frac{1}{2}} : x = \dfrac{1}{3\frac{1}{2}} : \dfrac{1}{2\frac{1}{2} + 3\frac{1}{2}}$.

(7) $\dfrac{1}{2 + \dfrac{1}{3\frac{1}{4}}} : \frac{13}{30} = x : 1$.

(8) $x : \dfrac{1}{4 + \dfrac{1}{5 + \dfrac{1}{6}}} = \dfrac{1}{6 + \dfrac{1}{5 + \dfrac{1}{4}}} : 21\frac{7}{10}$.

III. Place x in the first, second, third and fourth terms successively, and find its value in each case, the other three terms being:

(1) 12, 15, 20.

(2) 1, 2, 3.

(3) $3\frac{1}{3}$, $3\frac{3}{4}$, $2\frac{6}{7}$.

(4) $(\frac{1}{4} - \frac{1}{9})$, $(\frac{1}{2} + \frac{1}{3})$, $(\frac{1}{2} - \frac{1}{3})$.

Find x from $2\frac{4}{5} : 7\frac{1}{2} = 8\frac{3}{4} : x$;

$$\tfrac{14}{5} : \tfrac{15}{2} = \tfrac{35}{4} : x.$$

$$\begin{array}{r|l}
\times 5 & 14 : 5 \times \frac{15}{2} = \frac{35}{4} : x \\
\times 2 & 2 \times 14 : 5 \times 15 = \frac{35}{4} : x \\
\times 4 & 4 \times 2 \times 14 : 5 \times 15 = 35 : x \\
\div 7 & 4 \times 2 \times 2 : 5 \times 15 = 5 : x
\end{array}$$

$$x = \frac{5 \times 5 \times 15}{4 \times 2 \times 2} = \tfrac{375}{16} = 23\tfrac{7}{16}.$$

This example shews that any denominator may be transferred as a factor to a dissimilar term. Shortly, thus:

$$\begin{array}{llll}
\cancel{2\frac{4}{5}} : & \cancel{7\frac{1}{2}} = & \cancel{8\frac{3}{4}} : x & \\
\cancel{14} & 5 & \cancel{35} & \\
2 & 15 & 5 & \\
4 & & & x=\dfrac{5\times 5\times 15}{2\times 4\times 2}=23\frac{7}{16}. \\
2 & & &
\end{array}$$

§ 8. Examine the proportion £5 : £8 = £7 : x. Here we can attach no sense to the product of the means. (Part I. Ch. V. § 17.) The question is: Of what sum of money is £7 the same fraction that £5 is of £8? The concrete quantity £5 is the same fraction of the concrete £8 that the abstract number 5 is of 8, therefore substituting abstract for concrete numbers, the rules previously given become intelligible.

Find x from £1. 0*s.* $7\frac{1}{2}$*d.* : £1. 13*s.* = £1. 8*s.* 4*d.* : x

$$\begin{array}{lllll}
 & £1\frac{1}{32} & : £1\frac{13}{20} & = £1.\ 8s.\ 4d. : x & \\
\therefore & \frac{33}{32} & : \frac{33}{20} & & \\
 & \cancel{33} & \cancel{32} & & \\
 & \cancel{20} & \cancel{33} & & \\
 & 5 & 8 & & x=\frac{8}{5} \text{ of } £1.\ 8s.\ 4d.=£2.\ 5s.\ 4d.
\end{array}$$

or,

$$\begin{array}{llll}
\cancel{20\frac{5}{8}}s. & : \cancel{33}s. & = £1.\ 8s.\ 4d. : x & \\
\cancel{165} & 8 & & \\
\cancel{15} & \cancel{3} & & \\
5 & & & x=\frac{8}{5} \text{ of } £1.\ 8s.\ 4d. \text{ as before.}
\end{array}$$

or,

$$\begin{array}{llll}
\cancel{495} \text{ halfpence} & : \cancel{792} \text{ halfpence} & = £1.\ 8s.\ 4d. : x & \\
\cancel{55} & \cancel{88} & & \\
5 & 8 & & x=\frac{8}{5} \text{ of } £1.\ 8s.\ 4d. \text{ as before.}
\end{array}$$

We see that the choice of the unit is immaterial. It is best to choose the largest which is readily obvious.

§ 9. Find the cost of 35 articles at £1. 18*s.* $10\frac{1}{2}$*d.* for 21 articles.

The beginner will find it convenient first to arrange the problem thus:

Statement: 21 articles cost £1. 18*s.* $10\frac{1}{2}$*d.*
Question: 35 „ £x.

It is clear that the ratio of the two numbers of articles must equal that of the two prices, or that $21 : 35 = £1.\ 18s.\ 10\frac{1}{2}d. : x$. Hence $3 : 5$, &c.; $\therefore\ x = \frac{5 \times £1.\ 18s.\ 10\frac{1}{2}d.}{3} = £3.\ 4s.\ 9\frac{1}{2}d.$

If 14 articles cost £3. 17s. 7d., how many articles can be bought for £1. 10s. $5\frac{3}{4}d.$?

$$\begin{array}{lccc} \text{Art.} & £. & s. & d. \\ 14 \text{ cost} & 3 & 17 & 7 \\ x \quad ,, & 1 & 10 & 5\frac{3}{4} \end{array}$$

$$14 : x = £3\ 17\ \ 7 : £1\ 10\ \ 5\tfrac{3}{4}.$$

It is, however, usual to make x the fourth term; thus:

$$\begin{array}{llll} £3.\ 17s.\ 7d. & : £1.\ 10s.\ 5\frac{3}{4}d. & = \cancel{1}\cancel{4} & : x \\ \cancel{3}\cancel{7}\cancel{2}\cancel{4} \text{ farthings} & \cancel{1}\cancel{4}\cancel{6}\cancel{3} \text{ farthings} & \cancel{2} & \\ \cancel{5}\cancel{3}\cancel{2} & 209 & 1 & \\ \cancel{2}\cancel{6}\cancel{6} & & & \\ 38 & & & \end{array}$$

$$\frac{209 \times 1}{38} = \frac{209}{38} = 5\frac{\cancel{19}}{\cancel{38}}\frac{1}{2}.$$

Ans. $5\frac{1}{2}$ articles.

If the profits on £125. 14s. $9\frac{1}{4}d.$ be £6. 15s., what capital must I invest to get a profit of £168. 15s.?

$$\begin{array}{lllll} \text{Capital.} & & & \text{Profit.} & \\ £125\ 14\ \ 9\frac{1}{4} & \text{brings} & £6 & 15 & 0 \\ \quad x & \quad ,, & 168 & 15 & 0 \end{array}$$

$$\begin{array}{lll} £6.\ 15s.\ 0d. & : £168.\ 15s. & = £125.\ 14s.\ 9\frac{1}{4}d. : x \\ \cancel{1}\cancel{3}\cancel{5} \text{ shillings} & \cancel{3}\cancel{3}\cancel{7}\cancel{5} \text{ shillings} & \\ \cancel{2}\cancel{7} & 6\cancel{7}\cancel{5} & \\ \cancel{3} & \cancel{7}\cancel{5} & \\ 1 & 25 & \end{array}$$

$$£125.\ 14s.\ 9\tfrac{1}{4}d. \times 25 = £3143.\ 9s.\ 3\tfrac{1}{4}d.\ \textit{Ans.}$$

If $11\frac{1}{4}$ articles cost £2. 5s. 10d., what will $19\frac{11}{16}$ articles cost?

$$\begin{array}{lccc} \text{Art.} & £. & s. & d. \\ 11\frac{1}{4} \text{ cost} & 2 & 5 & 10 \\ 19\frac{11}{16}\ \ ,, & & x & \end{array}$$

$$\begin{array}{lll} \cancel{1}\cancel{1}\frac{1}{4} & : \cancel{1}\cancel{9}\cancel{\tfrac{11}{16}} & = £2.\ 5s.\ 10d. : x \\ \cancel{4}\cancel{5} & \quad \cancel{4} & \\ \cancel{1}\cancel{6} & \cancel{3}\cancel{1}\cancel{5} & \\ 4 & 6\cancel{3} & \\ \cancel{9} & \quad 7 & \end{array}$$

$$\frac{£2.\ 5s.\ 10d. \times 7}{4} = £4.\ 0s.\ 2\tfrac{1}{2}d.\ \textit{Ans.}$$

Find the cost of 1 ton, 15 cwt., 3 qrs., 5 lbs., if 13 cwt., 0 qrs., 21 lbs. cost £4. 16*s.* $6\frac{1}{2}$*d.*?

Tons.	cwt.	qrs.	lbs.		£.	s.	d.
0	13	0	21	cost	4	16	$6\frac{1}{2}$
1	15	3	5	,,		x	

Cwt. qrs. lbs.		Tons. cwt. qrs. lbs.		£. s. d.		
13 0 21	:	1 15 3 5	=	4 16 $6\frac{1}{2}$	:	x
1477 lbs.	:	4009 lbs.				

$$\frac{£4.\ 16s.\ 6\frac{1}{2}d. \times 4009}{1477} = £13.\ 2s.\ 0\tfrac{1}{2}d.\ \textit{Ans.}$$

How much spirits of wine can I get for 4*s.* $3\frac{3}{4}$*d.* at 23*s.* per gallon?

Gallon.	£.	s.	d.
1	1	3	0
x		4	$3\frac{3}{4}$

$$£1.\ 3s. : 4s.\ 3\tfrac{3}{4}d. = 1 : x$$

$$\cancel{23}\times\cancel{12}\times\cancel{4} : \cancel{207} = \cancel{4}\times\cancel{2} : x$$

$$\begin{array}{lll} 1 & \cancel{6} & \cancel{9} \\ & 2 & 3 \end{array} \qquad \tfrac{3}{2} = 1\tfrac{1}{2} \text{ pints. } \textit{Ans.}$$

Here £1. 3*s.* is expressed as $23 \times 12 \times 4$ farthings without multiplying out; 4*s.* $3\frac{3}{4}$*d.* are 207 farthings, and 1 gallon is 4×2 pints. It is best, where possible, merely to indicate operations, without performing them, as this facilitates cancelling. Thus here the one gallon of the third term is expressed as 4×2 pints, because the factors thus introduced cancel against factors of the first term.

Exercise XLVIII.

(1) Find the cost of 72 articles, if 40 cost £3. 12*s.* $8\frac{1}{2}$*d.*

(2) If 15 sacks of potatoes cost £4. 4*s.*, what shall we pay for 50 sacks?

(3) What is the cost of 104 yards at £1. 11*s.* 6*d.* for 91 yards?

(4) What will be the weekly keep of 195 horses, if 117 horses can be kept for £46. 16*s.* 9*d.*?

(5) What will be the carriage for 161 miles at 18*s.* 6*d.* for 276 miles?

(6) How many articles can be bought for £15 if 10 cost £25 ?

(7) If 15 sacks of potatoes cost £4. 4*s.*, how many can be bought for £2. 12*s.* 6*d.* ?

(8) How many yards can be bought for £33. 6*s.* 8*d.* at £100 for 300 yards ?

(9) If 45 yards of trench are dug by 15 men, how many men would be required to dig in the same time 63 yards ?

(10) If a railway ticket for 125 miles cost 7*s.* $9\frac{3}{4}d.$, how far ought I to be able to travel for £3. 2*s.* 6*d.* ?

(11) Find the cost of 4 tons, 17 cwt., 3 qrs., if 11 cwt., 2 qrs. cost 19*s.* $10\frac{1}{2}d.$?

(12) Gold costs £3. 17*s.* $10\frac{1}{2}d.$ per oz. troy. Find the value of 1 lb. av. ?

(13) How many oz. troy of gold shall I get for £103. 16*s.* 8*d.* ?

(14) Find the wages for 2 years, 8 months, at £10. 15*s.* per annum.

(15) A and B divide between them a hogshead of claret (23 dozen), costing £34. 10*s.* A takes 150 bottles. How much has B to pay ?

(16) If 3 qrs., 5 lbs. cost 3*s.* $8\frac{1}{2}d.$, what will $2\frac{1}{2}$ cwt. cost ?

(17) How long will £368. 19*s.* 1*d.* keep me, if £96. 0*s.* 7*d.* suffices from March 17 to June 20 ?

(18) How much old brandy can I get for £1. 17*s.* 6*d.* at £3 per gallon ?

(19) Taking the diameter of the earth at 7917 miles, and the highest mountain at 29,000 feet, by what fraction of an inch ought this mountain to be represented on a globe 1 yard in diameter ?

(20) If 6 cwt., 1 qr., 21 lbs., cost £5. 9*s.* $5\frac{1}{4}d.$, what weight can be bought for £17. 17*s.* ?

(21) $4\frac{7}{8}$ articles cost 9*s.* 10*d.* Find the cost of $10\frac{4}{5}$ articles.

(22) How many articles can be bought for 17*s.* $4\frac{1}{2}d.$, if $8\frac{4}{7}$ articles are bought for 2*s.* $2\frac{2}{5}d.$?

(23) A owns $\frac{5}{16}$ of a ship, B owns $\frac{5}{12}$ of it. If A's share is worth £2950, what is B's worth?

(24) If for £1700 I buy $\frac{9}{10}$ of a ship, what fraction of it can I buy for 1270 guineas?

(25) Find the fourth proportional to $4\frac{3}{8}$, $5\frac{2}{5}$, $7\frac{11}{12}$ [i.e. place x in the fourth term].

(26) The value of a fraction is $\frac{19}{20}$, the numerator is $33\frac{5}{8}$. What must the denominator be?

§ 10. Compare the following questions:

(1) If 8 articles cost £30, what will 12 cost?

(2) If 8 men do a piece of work in 30 days, how long will 12 men require?

In (1) more articles cost *more* money; in (2) more men require *less* time. Articles and their prices, horses and their provender, men and the work they can do, &c., are said to be in *direct* proportion, which means that twice as many articles cost twice as much money, three times as many articles cost three times as much, and so on, or twice as many horses require twice as much provender, &c. But men and the time required are in a different relation to each other; for twice as many men require half the time, three times the number of men require one-third of the time, and so on. Quantities thus related to each other are said to be in *inverse* proportion.

Direct.	Inverse.
(1) $\downarrow$ If 8 men earn £30, $\downarrow$	(2) $\uparrow$ If 8 men require 30 days, $\downarrow$
12 men will earn £x.	12 „ x days.

In (1) the number 30 is to be altered in the ratio of 8 : 12.
In (2) „ „ 12 : 8.

Hence the resulting proportions are:

(1) $8 : 12 = 30 : x$. *Ans.* £45. (2) $12 : 8 = 30 : x$. *Ans.* 20 days.

Before "stating" a proportion, we must always determine whether the problem is one of direct or of inverse proportion. The beginner will find it convenient to indicate his conclusion by arrows as above.

EXERCISE XLIX.

(1) If 8 men can mow a field in 5 days, how many men will mow it in 2 days?

(2) If 8 men can mow a field in 5 days, how many days will 10 men require?

(3) If we require $75\frac{3}{4}$ yds. of carpet $\frac{3}{4}$ yds. wide, to cover a room, how many yards of carpet $1\frac{1}{8}$ yds. wide, will be required?

(4) If £483 gain £27. 15*s.* interest in 1 year, how much capital will make the same profit in $7\frac{1}{2}$ months?

(5) If with a given sum of money I can buy 112 dozen at 7*s.* $4\frac{1}{2}$*d.* per dozen, how many articles can I buy at 10*s.* 6*d.* per dozen?

(6) If the wages of an Austrian workman be 1*s.* 8*d.* a day, and of a Lancashire workman 4*s.*, how many of the former can be hired for the wages of 375 of the latter?

(7) What sum of money will at $3\frac{1}{2}$ per cent. yield the same interest that 400 guineas yield at 5 per cent.?

(8) The capital of a company is raised by the issue of 1750 shares at £45 each. What would be the value of each share, if 3000 shares had been issued to raise the same amount?

(9) If 6352 stones, 3 ft. long, are required for a certain wall, how many stones, 2 ft. long, will be wanted?

(10) What length of land must be cut off from a piece $13\frac{1}{2}$ poles wide, to contain an area equal to a field 88 yds. long and 55 yds. broad?

(11) If $6\frac{3}{8}$ bushels last $1\frac{7}{8}$ days, how many days will $14\frac{2}{9}$ bushels last?

(12) If the daily allowance is $6\frac{3}{8}$ bushels, the store will last $1\frac{7}{8}$ days. How long will the store last with a daily allowance of $14\frac{2}{9}$ bushels?

§ 11. Chain Rule.

All questions of proportion can be stated in the shape of a fraction. It is convenient to draw the line perpendicularly. Find the cost of 45 articles if 20 cost £6. 6*s.* This question stated thus :

$$\begin{array}{r|l} \pounds x & 45 \text{ art.} \\ 20 \text{ art.} & \pounds 6.\ 6s. \end{array}$$

is read: "How many pounds for 45 articles if 20 articles cost £6. 6*s.*?" 45 × £6. 6*s.* is the numerator and 20 the denominator of the fraction sought. Cancelling, we obtain, 9 × £1. 11*s.* 6*d.* = £14. 3*s.* 6*d.* *Ans.* £14. 3*s.* 6*d.*

$$\begin{array}{r|lll} x & \cancel{45} & & \\ \cancel{20} & \cancel{6} & \cancel{6} & \cancel{0} \\ \cancel{4} & 9 & & \\ 1 & 1 & 11 & 6 \end{array}$$

The "chain" is correct for *direct* proportion (1) if each line begins with the denomination with which the previous line ended, (2) if the chain finishes with the same denomination with which it began, and (3) if the original sense of the question is preserved when read off as above.

This rule is applicable with advantage to such questions as the following: How many pounds of tea must be given for 28 lbs. of rice, if 4 lbs. of sugar are worth 1 lb. of coffee, 15 lbs. of sugar are worth 14 lbs. of rice, and 30 lbs. of coffee are worth 7 lbs. of tea?

Denominator.		*Numerator.*
tea	x	28 rice
rice	14	15 sugar
sugar	4	1 coffee
coffee	30	7 tea

This chain is stated correctly; for the second and third terms, the fourth and fifth terms, the sixth and seventh terms are respectively of like denomination, and so also are the first and last. Moreover, when read off, the statement preserves the original sense. The problem might be broken up into a number of separate questions of direct proportion. Thus :

How much sugar for 28 rice if 14 rice = 15 sugar? *Ans.* $\frac{28 \times 15}{14}$ sugar.

How much coffee for $\frac{28 \times 15}{14}$ sugar if 4 sugar = 1 coffee? *Ans.* $\frac{28 \times 15 \times 1}{14 \times 4}$ coffee.

How much tea for $\frac{28\times15\times1}{14\times4}$ coffee if 30 coffee $=7$ tea? *Ans.* $\frac{28\times15\times1\times7}{14\times4\times30}$ tea, which is the fraction obtained above in one statement.

$$\begin{array}{r|l} x & \cancel{28} \\ \cancel{14} & \cancel{15} \\ 4 & 1 \\ \cancel{30} & 7 \\ \cancel{2} & \cancel{2} \end{array} \qquad \tfrac{7}{4}=1\tfrac{3}{4}. \quad \textit{Ans. } 1\tfrac{3}{4} \text{ lbs. tea.}$$

Questions of this kind occur very rarely, but problems of "arbitration of exchange" are of daily occurrence, and are solved most readily by Chain Rule.

How many pounds sterling must be paid for 32830 francs, if 35 francs are worth 36 roubles; $4\frac{1}{5}$ Austrian florins $=2\frac{7}{10}$ Prussian thalers, 12 roubles $=5$ Austrian florins, and £1 $=6\frac{7}{10}$ thalers.

$$\begin{array}{ll|l} \text{£} & x & \cancel{32830} \text{ fr.} \\ \text{fr.} & \cancel{35} & \cancel{36} \text{ r.} \\ \text{r.} & \cancel{12} & 5 \text{ fl.} \\ \text{fl.} & \cancel{4\frac{1}{5}} & \cancel{2\frac{7}{10}} \text{ th.} \\ \text{th.} & \cancel{6\frac{7}{10}} & 1 \text{ £} \\ & \cancel{21} & \cancel{5} \\ & \cancel{10} & 27 \\ & \cancel{67} & \cancel{10} \\ & \cancel{7} & \cancel{3} \\ & \cancel{7} & \cancel{4690} \\ & & \cancel{670} \\ & & 10 \end{array} \qquad \begin{array}{c} 5\times27\times10=1350. \\ \textit{Ans. } \text{£}1350. \end{array}$$

Exercise L.

(1) Find the cost of 72 yds. at the rate of £5. 12*s.* 6*d.* for 10 yds.

(2) If $7\frac{1}{4}$ cwt. cost £26. 10*s.* 4*d.*, what will $43\frac{1}{2}$ cwt. cost?

(3) If 3 paces are equal to 2 yds., how many paces will there be in $106\frac{2}{3}$ yds.?

(4) If 5 balls of lead weigh as much as 9 of iron, and 7 of marble as much as 3 of iron, how many balls of marble will equal in weight 35 balls of lead?

(5) If 1 dollar is worth 4*s.* $1\frac{1}{2}$*d.* and also 5 fr., 17 centimes, find the value of a franc in sterling money (1 fr. $=100$ centimes).

(6) How many francs are worth £1, if $49\frac{1}{2}d.$ buy one dollar, and 100 dollars fetch 517 fr.?

(7) Find the value of the wool from 14,000,000 sheep, at £8. 16*s.* per cwt., if 11 sheep yield 25 lbs. of wool.

(8) How many francs for £1, if 35 Flemish shillings = £1, 480 rees are worth 3 francs and 400 rees are worth $3\frac{1}{2}$ Flemish shillings?

(9) If the rent of land in France be 140 francs per hectare, calculate the rate per acre in English money, 25 francs being equal to £1 and 100 hectares to 247 acres.

(10) If 1 metre = $39\frac{37}{100}$ in. and £1 = $25\frac{3}{8}$ fr., what is the cost in English money of 1 yd. at $1\frac{9}{20}$ fr. per metre?

(11) If $11\frac{11}{20}$ Dutch florins are given for $24\frac{9}{10}$ francs, 385 florins for 442 marks Hambro' and $68\frac{1}{4}$ marks for 32 Russian roubles, how much English money should be given for 2010 roubles at $25\frac{1}{8}$ fr. per £1?

(12) If 3 lbs. of tea are worth 4 lbs. of coffee, and 6 lbs. of coffee are worth 20 lbs. of sugar, how many lbs. of sugar can be had for 9 lbs. of tea?

(13) The day's journey in Turkey being 10 hours of $4\frac{1}{2}$ English miles each, and 11 English miles being equal to 12 Roman miles, how many Roman miles are there in 13 days' journey in Turkey?

(14) If £3 = 20 thalers, 25 gulden = 62 francs, 25 thalers = 93 francs; find how many gulden are equal to £1 sterling.

(15) If £1 = $25\frac{1}{2}$ francs, $9\frac{1}{3}$ florins = 20 francs, how many florins for £1?

(16) How much capillaire must be added to 580 gallons of dry gin, if to 100 gals. of gin is put 45 lbs. of sugar, and 1 gallon of capillaire has the sweetening power of 13 lbs. of sugar?

(17) Find the cost of 30 pieces of lead, each weighing 1 cwt., 12 lbs., at 16*s.* 4*d.* per cwt.

Exercise LI.

(1) A person takes 2 paces to walk 3 yds. How many yds. will he get over in 250 paces?

(2) How many paces will he take in a mile?

(3) The clothing of a regiment of 750 soldiers amounts to £2831. 5*s.* What will the clothing of 3500 men cost?

(4) A bankrupt owes £1954, his assets are £840. 12*s.* $6\frac{1}{2}$*d.* What will a creditor for £153 recover?

(5) The circumference of the earth at the equator is 24,900 miles. At what rate per hour is a person there carried round, one whole rotation being made in 23 h., 56 min.?

(6) If the 6*d.* loaf weighs 4 lbs. when flour costs £3. 5*s.* a sack, what ought it to weigh if flour cost £2. 15*s.* a sack?

(7) If the 4 lb. loaf cost 6*d.* when flour is £3. 5*s.* a sack, what ought it to cost when flour is £2. 15*s.* a sack?

(8) If the 6*d.* loaf weighs 4 lbs., what ought the $8\frac{1}{2}$*d.* loaf to weigh?

(9) If in a picture a tree 33 feet high is represented by a drawing $1\frac{1}{2}$ inches high, what should represent the height of a house 45 feet high?

(10) How high must a shrub be which is represented in the picture by $\frac{3}{7}$ inch?

(11) If a country 630 miles long is represented in a raised map by a length of $5\frac{1}{2}$ feet, how high ought a mountain of 15,750 feet to be represented on the map?

(12) If $1\frac{1}{2}$ inches represent the distance of the moon from the earth, how far off should the sun be placed, the actual distances being taken as 238,793 and 95,517,200 miles respectively?

(13) If $1\frac{1}{2}$ inches represent the distance of the sun from the earth, how far off should the nearest fixed star be placed, its distance being taken as 20,185,649,876,000 miles?

(14) If light takes 8 min., 13 secs. to travel from the sun to the earth, how long will it take from the moon to the earth, and how long from the star?

(15) The area of a certain garden is $1\frac{1}{2}$ acres, the width being 425 ft. What will its area be if the width be made 510 ft.?

(16) A certain garden is 440 feet long and 100 ft. broad. What would be the breadth of a garden of the same size whose length was 363 ft.?

(17) Find the cost of 7 tons, 12 cwt., 3 qrs., at £1. 10*s*. 10*d*. for 3 cwt., 1 qr.

(18) Find the cost of a tankard, weighing 1 lb., 7 oz., 14 dwts., at £1. 14*s*. 10*d*. for $5\frac{1}{2}$ oz.

(19) If the carriage of 15 tons, 15 cwt., cost £1. 12*s*. 6*d*., what will be the charge for carrying 3 tons, 17 cwt.?

(20) If 15*s*. 9*d*. pays the carriage on 2 tons, 17 cwt., what weight can be carried for a guinea?

(21) If goods are carried 45 miles for 7*s*. 6*d*., how far ought they to be carried for 10*s*. 6*d*.?

(22) If for a certain payment 15 tons, 10 cwt. are carried over 100 miles, how far ought 4 tons, 13 cwt. to be carried?

(23) If for a certain payment 10 tons, 9 cwt. are carried 150 miles, what weight ought to be carried over 200 miles?

(24) If from every £92. 10*s*. of capital I obtain £3 income, how much shall I obtain from £740?

(25) A bankrupt pays 9*s*. $4\frac{1}{2}$*d*. in the £. What will be lost on a debt of £324. 13*s*. 4*d*.?

(26) My gross income is £730. 15*s*. What will be left me after paying 5*d*. in the £ income-tax?

(27) Find a fourth proportional to:

a. 6, 9, 10.
b. 7, 8, 9.
c. $2\frac{1}{2}$, $3\frac{1}{3}$, $1\frac{7}{8}$.
d. $\frac{1}{7}$ of $\frac{1}{3}$, $\frac{1}{4}$ of $\frac{1}{5}$, $\frac{1}{7}$ of $\frac{1}{8}$.
e. $(\frac{2}{3}+\frac{1}{2})$, $(\frac{2}{3}-\frac{1}{2})$, $\frac{2}{3}$ of $\frac{1}{2}$.
f. $\frac{1}{2\frac{1}{2}}$, $\frac{1}{3\frac{1}{2}}$, $\frac{1}{4\frac{1}{2}}$.

(28) Find the cost of 2 tons, 7 cwt., 3 qrs., at £5. 18*s*. for 15 cwt., 1 qr.

(29) Find the cost of 73 yds., 1 qr., 3 nls., at £39. 11*s*. 8*d*. for 1000 yds.

(30) If $1\frac{7}{8}$ articles cost £$5\frac{1}{16}$, how much will $2\frac{13}{16}$ articles cost?

(31) If £$\frac{2}{7}$ buy $\frac{5}{7}$ of 1 article, what will £$2\frac{2}{5}$ buy?

(32) A train goes from London to Bristol in $3\frac{1}{3}$ hours, travelling at the rate of 3 miles in 5 minutes. (1) How far is it to Bristol?

(2) How long would it take to Exeter (194 miles) at the same rate? (3) How long will it take to Bristol and to Exeter if the rate be increased to 40 miles an hour?

(33) If 770 gallons of creosote have the heating power of $8\frac{3}{4}$ tons of coal, how many gallons a day would be required for a steamer which consumes 50 tons of coal daily?

(34) Find the cost of 1 lb. avoirdupois of gold, at £3. 17*s.* $10\frac{1}{2}$*d.* per oz. troy.

(35) Find the weight of 4361 sovereigns in av. weight at the same rate.

(36) 10 cubic inches of gold weigh as much as 193 cubic inches of water. What is the size of a nugget weighing as much as a cubic foot of water?

(37) If $4\frac{1}{2}$ tons of coal fill a cellar 9 ft. long, 5 ft. broad, 5 ft. high, what space will be required for the coal of a steamer carrying 3 weeks' consumption, at 20 tons per day?

(38) The space between the freezing and the boiling points of water is divided into 180, 80, 100 degrees respectively on Fahrenheit's, Réaumur's and the Centigrade thermometers. How many degrees of the second and third are equivalent to 18, 27, 45 and 63 degrees on the first?

(39) In Fahrenheit's thermometer freezing point is marked 32^0; on the others, zero; so that 32^0 F $= 0^0$ C $= 0^0$ R. Translate the following readings into readings of the other two: 41^0 F; 8^0 R; 40^0 C; 40^0 R; 90^0 C; 86^0 F; 0^0 C; 100^0 C; 100^0 F; 50^0 C; 50^0 F; 50^0 R.

(40) 90 degrees are 100 grades. How many grades are equal to $65\frac{5}{8}$ degrees, and how many degrees are equal to $65\frac{5}{8}$ grades?

(41) If 600 men can dig a cutting 750 yards long in 23 days, how long would 460 men take?

(42) How long would the 600 men take if the cutting were 800 yards long?

(43) What part of the 800 yards would $\frac{4}{5}$ of the number of men do?

(44) How long would $\frac{4}{5}$ of the men take to do the 800 yards?

(45) How many men would it take to do the 750 yards in $\frac{4}{5}$ of the time?

(46) If 24 men dig a ditch in 3 days, how long would they take to dig a ditch half as long again, half as deep again, and half as broad again?

(47) How many men would it take to dig the second ditch in the same time?

(48) What will be my new expenditure, supposing it to have been originally 300 guineas, if I alter it in the ratio of 7 to 12?

(49) If brickwork, 84 ft. high, $73\frac{1}{2}$ ft. long, cost £700, how must I reduce the thickness so that it may cost only £600? What reduction of height would have given me the same result? What would be the cost if both height and length were thus reduced?

(50) If 3 cwt., 2 qrs., 12 lbs. cost £9, what is the price of 6 lbs.?

(51) A bankrupt's debts amount to £4586. 8*s.*, and his effects to 3822 guineas. How much will a creditor receive on a debt of £700?

(52) A railway train travels $\frac{1}{4}$ of a mile in 18 seconds. How many miles an hour does it travel at this rate?

(53) If $35\frac{1}{2}$ lbs. of sugar cost £1. 2*s.* $2\frac{1}{4}$*d.*, how much will 2 cwt., 1 qr. 23 lbs. cost?

(54) If £100 put out to interest for 9 months becomes £103, what sum would amount to £193. 2*s.* 6*d.* in the same time?

(55) If $\frac{1}{143}$ of $3\frac{2}{3}$ of $\frac{7}{9}$ of $5\frac{1}{5}$ of 22 lbs. of sugar cost $8\frac{1}{4}$*d.*, how much will 1 ton, 11 cwt., 3 qrs. cost?

(56) If $\frac{3}{11}$ of $\dfrac{3}{2\frac{1}{2}}$ of $\frac{5}{27}$ of a ship cost £3710, what part of the ship can be bought for £100?

(57) If a garrison of 1500 men have provisions for 13 months, how long will their provisions last if it be increased by 700 men?

(58) If a man's step be 2 ft., 4 in., and a horse's 2 ft., 9 in., how many steps of the horse are equal to 108 steps of the man?

(59) If 432 and 750 be two among several factors of a number, what must I substitute for the latter if the former be altered to 540, in order that the final product may remain the same?

(60) A coat requiring $2\frac{3}{4}$ yards of cloth, $1\frac{1}{4}$ yds. wide, is lined with silk 39 in. wide. How many yards of silk will be required?

(61) If the quotient be altered from 730 to 219, what must the dividend be, which was before 1,000,000, so that the divisor may be unaltered?

(62) If the dividend be altered from 999999 to 285714, what will be the ratio of the former quotient to the new one, the divisor being unaltered?

(63) A bankrupt owes me £721. 12*s.* 6*d.*, and pays me only £240. 10*s.* 10*d.* How much ought he to pay another creditor, whose claim amounts to £312. 14*s.* 1*d.*?

(64) 779 cubic inches of cast iron weigh as much as 725 cubic inches of forged iron. What bulk of forged iron will weigh as much as 1000 cubic feet of cast iron?

§ 11. THE UNITARY METHOD *(continued).*

If 2 tons, 5 cwt. can be carried over 150 miles for 14*s.* 6*d.*, how far should 8 tons, 15 cwt. be carried for £1. 4*s.* 2*d.*?

Statement: 2 tons, 5 cwt. can be carried for 14*s.* 6*d.* over 150 miles.
Question: 8 tons, 15 cwt. " £1. 4*s.* 2*d.* " x "

Mod. op.:

2 tons, 5 cwt. (45 cwt.) can be carried for 14*s.* 6*d.* (174*d.*) ......over 150 miles.
1 cwt. " " ......45×150 "
8 tons, 15 cwt. (175 cwt.) " " $\frac{1}{175}$ of 45×150 "
" " 1*d.* $\frac{1}{174}$ of $\frac{1}{175}$ of 45×150 "
" " £1. 4*s.* 2*d.* (290*d.*) $\frac{290}{174}$ of $\frac{1}{175}$ of 45×150 "

$$\frac{\overset{\overset{\overset{1}{\cancel{29}}}{\cancel{145}}}{\cancel{290}}}{\underset{\underset{\cancel{3}}{\cancel{87}}}{\cancel{174}}} \times \frac{\overset{\overset{3}{\cancel{15}}}{\cancel{45}}}{\underset{\underset{7}{\cancel{35}}}{\cancel{175}}} \times 150 = \frac{450}{7} = 64\tfrac{2}{7}.$$

Ans. $64\frac{2}{7}$ miles.

or thus:

2 tons, 5 cwt. (45 cwt.) can be carried for 14*s.* 6*d.* (174*d.*) .... over 150 miles.
1 cwt. " " .... 45×150 "
1 cwt. " 1*d.* .... $\frac{1}{174}$ of 45×150 "
8 tons, 15 cwt. (175 cwt.) " " ....$\frac{1}{175}$ of $\frac{1}{174}$ of 45×150 "
" " £1. 4*s.* 2*d.* (290*d.*) $290 \times \frac{1}{175}$ of $\frac{1}{174}$ of 45×150 "

$290 \times \frac{1}{175} \times \frac{1}{174} \times 45 \times 150 = 64\frac{2}{7}$, as before.

The second of these methods, in which all the quantities of the statement but the last are first reduced to *unity*, is the ordinary one. The reasoning is very obvious; but after working thus a few questions of Exercise LII., the more concise reasoning of Compound Proportion will be preferred.

§ 12. COMPOUND PROPORTION.

If 20 men earn £15 in 4 days, how many men will earn £24 in 6 days?

Here the result is affected not only by the alteration of the amount earned, but also by that of the number of days. The problem may be resolved into two separate proportions thus:

(1) If 20 men earn £15 in 4 days,
x „ £24 in the same time.

$$15 : 24 = 20 : x. \qquad \textit{Ans.}\ \frac{20 \times 24}{18}\ \text{men.}$$

(2) If $\frac{20 \times 24}{15}$ men earn £24 in 4 days,
x „ the same in 6 days.

$$6 : 4 = \frac{20 \times 4}{15} : x.$$

$$\textit{Ans.}\ \frac{\overset{4}{\cancel{20}} \times \overset{4}{\cancel{24}} \times 4}{\underset{3}{\cancel{15}} \times \underset{1}{\cancel{6}}} = \frac{64}{3} : 21\tfrac{1}{3}\ \text{men.*}$$

The same resulting fraction, $\frac{20 \times 24 \times 4}{15 \times 6}$, will evidently be obtained by the following method. Statement:

Men.	£.	Days.
20	15	4
x	24	6

$$\left.\begin{array}{r} 15 : 24 \\ 6 : 4 \end{array}\right\} = 20 : x$$

$$\textit{Ans.}\ \frac{20 \times 24 \times 4}{15 \times 6}\ \text{as above.}$$

* The interpretation that must be given to such an answer as this is, that £24 will pay for 6 days the wages of 21 men, with enough to spare to pay the third part of the wages of another man.

Hence each pair of numbers of like denomination must be separately considered with reference to that pair in which the x occurs, marking each direct proportion by a downward, and each inverse proportion by an upward arrow.

If 9 men can in 10 days, of 12 hours each, dig a trench 45 yards long, $1\frac{1}{4}$ ft. wide, and $1\frac{7}{8}$ ft. deep, how many hours a day must 8 men work to dig a trench $4\frac{7}{8}$ yds. long, $2\frac{1}{2}$ ft. wide, and 5 ft. deep, in 13 days?

Men.	Days.	Hours.	Yds. long.	Ft. wide.	Ft. deep.
$\uparrow$ 9	$\uparrow$ 10	$\downarrow$ 12	$\downarrow$ 45	$\downarrow$ $1\frac{1}{4}$	$\downarrow$ $1\frac{7}{8}$
8	13	x	$4\frac{7}{8}$	$2\frac{1}{2}$	5

1st. Put a downward arrow to the pair containing x.

2nd. Consider the men: more men, less hours. Inverse. $\therefore$ place to the men an upward arrow.

3rd. Consider the days: more days, less hours. Inverse. Arrow upwards.

4th. More length, more hours. Direct. Arrow downwards.

5th. More width, more hours. Direct. Arrow downwards.

6th. More depth, more hours. Direct. Arrow downwards.

Now "compound" the several proportion sums as follows:

$$\left.\begin{array}{rcl} \cancel{8} & : & \cancel{9} \\ \cancel{13} & : & \cancel{10} \\ \cancel{45} & : & \cancel{4\tfrac{7}{8}} \\ \cancel{1\tfrac{1}{4}} & : & \cancel{2\tfrac{1}{2}} \\ \cancel{1\tfrac{7}{8}} & : & \cancel{5} \end{array}\right\} = \underset{6}{\cancel{12}} : x$$

$$\begin{array}{rr} \cancel{8} & \cancel{39} \\ \cancel{2} & \cancel{5} \\ \cancel{5} & \cancel{4} \\ \cancel{15} & \cancel{8} \\ \cancel{5} & \cancel{3} \\ \cancel{2} & \cancel{5} \\ \cancel{5} & \end{array}$$

Ans. 6 hours.

The justification of this method of compounding will perhaps be rendered more evident to beginners by the use of the language of Fractions.

9 men require 12 hours, $\therefore$ 8 men require $\frac{9}{8}$ of 12 hours.

Again, for 13 instead of 10 days, they require $\frac{10}{13}$ of $\frac{9}{8}$ of 12 hours.

For $4\frac{7}{8}$ yds. instead of 45 yds. length, they require $\frac{4\frac{7}{8}}{45}$ of $\frac{10}{13}$ of $\frac{9}{8}$ of 12 hours.

For $2\frac{1}{2}$ ft. instead of $1\frac{1}{4}$ ft. width, they require $\frac{2\frac{1}{2}}{1\frac{1}{4}}$ of $\frac{4\frac{7}{8}}{45}$ of $\frac{10}{13}$ of $\frac{9}{8}$ of 12 hours.

And lastly, for 5 ft. instead of $1\frac{7}{8}$ ft. depth, they require $\frac{5}{1\frac{7}{8}}$ of $\frac{2\frac{1}{2}}{1\frac{1}{4}}$ of $\frac{4\frac{7}{8}}{45}$ of $\frac{10}{13}$ of $\frac{9}{8}$ of 12 hours.

The only difference between this result and that previously obtained is, that the numerators here were in that case means, and the denominators extremes. Every pair may thus be looked upon as a fraction, and in each case we have to determine which is the numerator.

Any factor common to both members of the pair may be struck out before commencing operations.

If I travel 300 miles in 6 days of 8 hours each, in how many days of 10 hours shall I travel 450 miles, travelling half as fast again?

Miles.	Days.	Hours.	Speed.
↓ $\cancel{300}$ 2	↓ 6	↑ $\cancel{8}$ 4	↑ 1
$\cancel{450}$ 3	x	$\cancel{10}$ 5	$1\frac{1}{2}$

$$\left.\begin{array}{rcl} \cancel{2} & : & \cancel{3} \\ 5 & : & 4 \\ \cancel{1\frac{1}{2}} & : & 1 \\ \cancel{3} & & \cancel{2} \end{array}\right\} = 6 : x.$$

$$\frac{6\times 4}{5} = \frac{24}{5} = 4\tfrac{4}{5}.$$

Ans. $4\frac{4}{5}$ days.

Exercise LII.

(1) If £240 be the wages of 6 men for 21 weeks, what will 28 men earn in 23 weeks?

(2) How many men can I employ for 7 months for £453. 12*s.*, if the wages of 50 men for 12 months is £1080?

(3) If I pay £$1\frac{3}{4}$ to 13 men for $3\frac{3}{4}$ days, what will be the wages of 30 men for $10\frac{2}{5}$ days?

(4) If 36 persons consume 240 pecks of corn in 30 days, how long will 100 pecks last 90 persons?

(5) If 15 pumps, working 8 hours a day, can raise 1260 tons of water in 7 days, how many pumps, working 12 hours a day, will be required to raise 7560 tons of water in 14 days?

(6) If 24,000 yds. of cotton cloth, $1\frac{3}{4}$ yds. wide, be worth £400 when raw cotton is at $4\frac{1}{2}d.$ per lb., what is the value of 12,600 yds. of cotton cloth, $1\frac{1}{4}$ yds. wide, when raw cotton is at $9\frac{9}{16}d.$ per lb., supposing the cost of manufacture to have risen in equal proportion?

(7) If the corn of 13 horses for 63 days cost £17. 6*s.* 8*d.* when corn is 4*s.* per bushel, how many horses will in 56 days consume corn to the value of £10. 13*s.* 4*d.* when corn is 4*s.* 6*d.* per bushel?

(8) If 48 pioneers in 5 days, of $12\frac{1}{2}$ hours each, can dig a trench $139\frac{3}{4}$ yds. long, $4\frac{1}{2}$ yds. wide and $2\frac{1}{2}$ yds. deep, how many hours per day must 360 pioneers work during 42 days to dig a trench $4910\frac{1}{16}$ yds. long, $4\frac{7}{8}$ yds. wide and $3\frac{1}{8}$ yds. deep?

(9) If 6 men can perform a piece of work in 12 days of 10 hours each, how many men will perform a piece of work four times as large in a fifth of the time, if they work the same number of hours a day, supposing that 2 of the second set can do as much work in an hour as 3 of the first set?

(10) What is the interest on £100 for 1 year, if the interest on £1303. 6*s.* 8*d.* for 10 years is £488. 15*s.*?

(11) If a block of marble be worth £50, what will be the value of a block twice as long, 3 times as broad, and $\frac{1}{4}$ of the thickness?

(12) If a cubic block cost £50, what would be the price of a cubic block 3 times as long?

(13) If a block of marble weigh 50 cwt., what is the weight of a block of iron, the dimensions of which are to those of the former in the proportion of 2 : 3, 3 : 4, and 4 : 5 respectively, weights of equal blocks of iron and marble being in the proportion of 77 : 27.

§ 13. Proportional Parts.

Break up a given quantity into two parts whose ratio shall be that of the numbers 2 and 3. The two parts are $\frac{2}{5}$ and $\frac{3}{5}$ of the given quantity, for these added together give the whole quantity,

and they are in the required ratio. Similarly, to break up a quantity into three parts whose ratios to one another are those of the numbers 4, 5, 6 (which is expressed by 4 : 5 : 6), break up the whole into $4+5+6=15$ parts, and take 4, 5 and 6 such parts severally. *Ans* $\frac{4}{15}$, $\frac{5}{15}$ and $\frac{6}{15}$, or $\frac{4}{15}$, $\frac{1}{3}$ and $\frac{2}{5}$ of the given quantity.

Generally: Divide the given quantity by the sum of the given numbers, and multiply the quotient by each number separately.

The ratios of two or more numbers are not altered if they are all multiplied or divided by the same number (Cf. § 6); thus $2:3:4 = 20:30:40$, and $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} = 15:10:6$ (each term being multiplied by 30). The general rule for simplifying ratios is to divide the given numbers by their G.C.M. if they are integers, or to multiply them all by the L.C.M. of their denominators if they are fractions. This will evidently reduce them to the simplest integral form.

Divide £10. 10s. into 4 parts, having the ratios 4 : 7 : 8 : 9.

$4+7+8+9=28$. The fractions are $\frac{4}{28}$, $\frac{7}{28}$, $\frac{8}{28}$, $\frac{9}{28}$ of £10. 10s.

$\frac{1}{28}$ of £10. 10s.	=	0	7	6
$\frac{4}{28}$,,	=	1	10	0
$\frac{7}{28}$,,	=	2	12	6
$\frac{8}{28}$,,	=	3	0	0
$\frac{9}{28}$,,	=	3	7	6
		£10	10	0

Ans. £1. 10s.; £2. 12s. 6d.; £3; £3. 7s. 6d.

N.B. It is well always to verify the separate results by addition. They should of course give the original number to be divided.

If three persons invest £200, £350 and £750 respectively in the purchase of a property which yields an income of £109. 4s., how should this be divided among them?

$200:350:750=4:7:15$. The shares should be $\frac{4}{26}$, $\frac{7}{26}$ and $\frac{15}{26}$ of £109. 4s.

$\frac{1}{26}$ of £109. 4s.	=	£4	4	0
$\frac{4}{26}$,,	=	16	16	0
$\frac{7}{26}$,,	=	29	8	0
$\frac{15}{26}$,,	=	63	0	0
		£109	4	0

Ans. £16. 16s.; £29. 8s.; £63.

If A invest £300 for 4 months, B £150 for 7 months, and C £200 for 9 months, how should the profit be divided?

If we take as a unit the investment of £1 for 1 month, A has invested 300 × 4 such units, B 150 × 7 units, and C 200 × 9 units.

$$\begin{aligned} &\cancel{300}\times 4 : \cancel{150}\times 7 : \cancel{200}\times 9\\ =\ &\cancel{6}\ \times 4 : \ \cancel{3}\ \times 7 : \ 4\ \times \cancel{9}\\ =\ &8 \quad : 7 \quad : 12, \end{aligned}$$

and the shares are $\frac{8}{27}$, $\frac{7}{27}$ and $\frac{12}{27}$ of the profit.

The sum of £15 was to be divided among the three head boys of a class in proportion to their marks. They obtained respectively $\frac{3}{4}$, $\frac{2}{3}$ and $\frac{1}{2}$ of the total number. What should each get?

$\frac{3}{4} : \frac{2}{3} : \frac{1}{2} = 9 : 8 : 6$, $\therefore$ $\frac{9}{23}$, $\frac{8}{23}$ and $\frac{6}{23}$ of £15 are the respective shares. *Ans.* £5. 17*s.* $4\frac{16}{23}$*d.*; £5. 4*s.* $4\frac{4}{23}$*d.*; £3. 18*s.* $3\frac{3}{23}$*d.*

Exercise LIII.

(1) Divide £3. 10*s.* into two parts which shall have the ratio of 5 : 7.

(2) Divide a guinea into three parts in the ratio of 2 : 3 : 4.

(3) Divide a guinea into 6 parts which shall have the ratios of the first 6 natural numbers.

(4) Two partners engage in business with capitals of £7000 and £9000 respectively, the profits amounting to £2400 a year. What should each receive?

(5) What is the value of the gold in a chain weighing 3 oz., 4 dwt. troy, supposing it to be 18 carats fine (i.e. 18 parts pure gold out of 24) at £3. 17*s.* $10\frac{1}{2}$*d.* per oz.

(6) If two partners engage in business, investing respectively £482. 1*s.* 8*d.* and £630. 8*s.* 4*d.*, what should each have of a profit of £51. 17*s.* 6*d.*?

(7) Four persons, A, B, C, D, rent a pasture for £57; A put in 8 cattle, B 9, C 10, and D 11. How much should each pay for his share?

(8) A tax of £489. 17*s.* is to be raised from 3 towns, the populations of which are respectively 2500, 3000 and 4200. How much should each town pay?

(9) 40 gallons of alcohol are mixed with 14 gallons of water, What weight of alcohol is there in every lb. weight of the mixture, the weights of equal measures of alcohol and water being in the ratio 4 : 5.

(10) Copper is $8\frac{9}{10}$, tin $7\frac{3}{10}$ as heavy as water. If 20 cubic inches of tin be mixed with 30 inches of copper, how many times its own weight of water will the mixture weigh?

(11) Four merchants, A, B, C, D, trade together. A's capital of £800 was in trade 8 months; B's of £700, 12 months; D's of £400, 6 months; and C's of £135, 4 months. What share of the profit should each receive?

(12) If 200 oz. of gold, 18 carats fine, are mixed with 128 oz., 15 carats fine, what is the weight of gold in the mixture?

(13) What is the fineness?

(14) Gunpowder contains $\frac{3}{4}$ of its weight of saltpetre; saltpetre is composed of 39 parts by weight of potassium, 14 of nitrogen and 48 of oxygen. How many lbs. of potassium are there in 909 lbs. of gunpowder?

(15) Divide £250 among A, B, C and D, so that A's share shall be to B's as 4 to 5, B's to C's as 5 to 6, C's to D's as 6 to 7.

(16) Divide £1000 among A, B, C and D, so that A's share shall be to B's as $\frac{1}{2}$ to $\frac{1}{3}$, B's to C's as $\frac{1}{3}$ to $\frac{1}{4}$, C's to D's as $\frac{1}{4}$ to $\frac{1}{5}$.

(17) Divide £31. 12*s.* 6*d.* among A, B, C and D, so that A's share shall be to B's as 2 to 3, B's to C's as 5 to 6, C's to D's as 8 to 9.

(18) Divide £6045 among A, B, C and D, so that A's share shall be to B's as $\frac{1}{2}$ to $\frac{2}{5}$, B's to C's as $\frac{5}{6}$ to $\frac{4}{9}$, and C's to D's as $\frac{7}{10}$ to $\frac{1}{8}$.

PART III.

APPROXIMATE CALCULATIONS.

ARITHMETIC.

PART III. APPROXIMATE CALCULATIONS.

CHAPTER I.

CONVERGING FRACTIONS.

§ 1. INSPECTING the answers to the majority of questions in Practice, Proportion, &c., where the data are not "cooked" but taken at random as they occur in the affairs of life, we find Fractions obtained with considerable labour, which are needlessly accurate. For example, on p. 65, the price of a certain quantity of goods is found to be £30. 1*s*. $5\frac{21}{320}$*d*. The most laborious part of the process consisted in finding the fraction $\frac{21}{320}$ of a penny, which, after all, cannot be paid, and is consequently disregarded. In the previous problem the result was £74. 13*s*. $2\frac{4}{7}$*d*. $\frac{4}{7}$ of a penny is more than $\frac{1}{2}$*d*. and less than $\frac{3}{4}$*d*.; it lies between these two values, but nearer to the lower; £74. 13*s*. $2\frac{1}{2}$*d*. is thus the nearest payable sum.

We may, however, easily imagine cases where the fraction could not be totally rejected, but where, nevertheless, another simpler fraction *tolerably near* the truth would sufficiently answer our purpose. The process by which we find simple expressions whose values are nearly equal to that of a more complicated one is called "Approximation."

§ 2. Examine the fraction $\frac{21}{320}$.

It is less than $\frac{1}{10}$, for $\frac{1}{10} = \frac{32}{320}$,

" $\frac{1}{12}$, for $\frac{1}{12} = \frac{26\frac{2}{3}}{320}$,

" $\frac{1}{15}$, for $\frac{1}{15} = \frac{21\frac{1}{3}}{320}$,

but it is greater than $\frac{1}{16}$, for $\frac{1}{16} = \frac{20}{320}$.

Therefore the fraction, which we will call "The Truth," lies between $\frac{1}{15}$ and $\frac{1}{16}$.

These two fractions only differ from one another by $\frac{1}{240}$; therefore neither of them differs from the truth which lies between them by so much as $\frac{1}{240}$, and each of them conveys an easier notion of the value of the fraction than does the original form. For many purposes either answer is a sufficiently close approximation; but the question may be such that $\frac{1}{240}$ is too large an error.

Assume two lines, A and B, being respectively 21 and 320 inches long. What fraction of B is A? *Ans.* $\frac{21}{320}$, which is at lowest terms, i.e. in the *simplest* form which is absolutely accurate. Dividing, nevertheless, both terms by 21, we obtain $\frac{1}{15\frac{5}{21}}$. Rejecting, for the sake of clearness, the $\frac{5}{21}$ in the denominator, we obtain $\frac{1}{15}$. This fraction must be an over-estimate, the denominator being too small. To approximate more closely to the truth, we may treat $\frac{5}{21}$ in a similar manner by dividing both terms by 5; $\frac{5}{21} = \frac{1}{4\frac{1}{5}}$, and we now obtain $\cfrac{1}{15+\cfrac{1}{4+\frac{1}{5}}}$.

Again rejecting the $\frac{1}{5}$ of the second denominator, we obtain $\frac{1}{15+\frac{1}{4}} = \frac{4}{61}$. This must be an under-estimate, the second denominator being too small; and therefore, the fraction $\frac{1}{4}$ being too large, $15\frac{1}{4}$ is too great a denominator of the whole fraction, i.e. the whole fraction $\frac{1}{15+\frac{1}{4}}$ is too small. The truth then lies between $\frac{1}{15}$ and $\frac{4}{61}$, and therefore does not differ from either by so much as $\frac{1}{15} - \frac{4}{61}$ or $\frac{1}{15 \times 61} = \frac{1}{915}$.

The numerator of the fraction now rejected being 1, the process does not admit of a repetition; and if this fraction be taken into account, the fraction $\cfrac{1}{15+\cfrac{1}{4+\frac{1}{5}}}$ will yield $\frac{21}{320}$, the original quantity.

We have thus obtained the following approximations:

First value	$\frac{1}{15}$ an over-estimate,	}	Limit of error,
Second „	$\frac{4}{61}$ an under-estimate,	}	$\frac{1}{15} - \frac{4}{61} = \frac{1}{915}$.
Third „	$\frac{21}{320}$ the truth.		

Take $\frac{99}{229}$. $\frac{99}{229} = \frac{99 \div 99}{229 \div 99} = \frac{1}{2 + \frac{31}{99}}$.

First approximation, $\frac{1}{2}$ an over-estimate.

$\frac{31}{99} = \frac{31 \div 31}{99 \div 31} = \frac{1}{3 + \frac{6}{31}}$.

Second approximation, $\frac{1}{2 + \frac{1}{3}} = \frac{3}{7}$ an under-estimate.

Limit of error, $\frac{1}{2} - \frac{3}{7} = \frac{1}{14}$.

$\frac{6}{31} = \frac{6 \div 6}{31 \div 6} = \frac{1}{5 + \frac{1}{6}}$.

Third approximation, $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5}}} = \frac{16}{37}$ an over-estimate.

Limit of error, $\frac{16}{37} - \frac{3}{7} = \frac{1}{259}$.

Fourth approximation, $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{6}}}} = \frac{99}{229}$ the truth.

Examination of the several steps will shew that the successive alternate divisions are in fact nothing but the several steps of the process for finding G.C.M. by the *third method*, Part I. p. 140, the quotients being taken as the successive denominators.

Mod. op.:

$\frac{99}{229}$

2, 3, 5, 6	
99	229
6	31
—	1

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{6}}}}$$

Find a series of approximations or convergents to the fraction $\frac{2877}{7518}$.

2, 1, 1, 1, 1, 2, 2, 4	
2877	7518
1113	1764
462	651
84	189
—	21

$$\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{4}}}}}}}}$$

			Difference between two successive estimates; hence limits of error.
First approximation,	$\frac{1}{2}$	over-estimate	
Second „	$\frac{1}{3}$	under „	$\frac{1}{6}$
Third „	$\frac{2}{5}$	over „	$\frac{1}{15}$
Fourth „	$\frac{3}{8}$	under „	$\frac{1}{40}$
Fifth „	$\frac{5}{13}$	over „	$\frac{1}{104}$
Sixth „	$\frac{13}{34}$	under „	$\frac{1}{442}$
Seventh „	$\frac{31}{81}$	over „	$\frac{1}{2754}$
Last „	$\frac{137}{358}$	the truth	

It will be observed that this last estimate does not bring back the form of the original fraction, because it was not at lowest terms. Dividing both terms of $\frac{2877}{7518}$ by 21, their G. C. M., the fraction $\frac{137}{358}$ will be obtained.

The alternation of values follows from the following considerations:

1. An increase of the numerator increases the value of the fraction.
2. An increase of the denominator decreases its value.
3. A denominator of a denominator is a numerator; but
4. The denominator of *that* is a denominator again; and so on.

Exercise I.

Find convergents and limits of error to the values of the following fractions:

(1) $\frac{181}{213}$. (2) $\frac{111}{235}$. (3) $\frac{417}{1002}$.

(4) Reduce the fraction that 1 kilometre (39,370 inches) is of an English mile to a continued fraction, and find 6 convergents.

(5) Find 4 convergents to the ratio of the diameter of a circle to its circumference 100000 : 314159.

(6) Find 5 convergents to the ratio of a lb. troy to a lb. av.

(7) Find 3 convergents to the ratio of a yard to a metre 360000 : 393708.

(8) Find 2 convergents to the ratio of a hectare ($11960\frac{1}{3}$ square yds.) to a square mile.

(9) Mont Blanc is 15,784 ft. high, and the diameter of the earth is $7912\frac{2}{5}$ miles. By what *aliquot* fraction of an inch would its height be nearly represented on a globe 18 inches in diameter?

CHAPTER II.

DECIMALS.

§ 1. We have found that the chief difficulty in fractional calculations is due to our having to manipulate fractions with different denominators. Fractions with denominators either the same or easily interconvertible present much less difficulty. The denominators 10, 100, 1000, &c., are easiest of conversion. Thus $\frac{1}{10} = \frac{10}{100} = \frac{100}{1000}$, &c.; $\frac{27}{100} = \frac{270}{1000} = \frac{2700}{10000}$, &c.

Learn by heart: *A fraction whose denominator is a power of* 10 *is called a Decimal Fraction; others are called Vulgar Fractions.*

§ 2. Decimalization of Vulgar Fractions.

Verification.

$\frac{1}{2} = \frac{1}{2}$ of $\frac{10}{10} = \frac{5}{10}$ .... $\frac{5}{10}\big|\frac{1}{2}$

$\frac{1}{4} = \frac{1}{4}$ of $\frac{10}{10} = \frac{2}{10}$ and $\frac{2}{10}$ over; $\frac{2}{10} = \frac{20}{100}$, $\frac{1}{4}$ of $\frac{20}{100} = \frac{5}{100}$, $\therefore \frac{1}{4} = \frac{2}{10} + \frac{5}{100} = \frac{20+5}{100} = \frac{25}{100}$ ...... $\frac{25}{100}\big|\frac{5}{20}\big|\frac{1}{4}$

$\frac{3}{4} = \frac{1}{4}$ of $3 = \frac{1}{4}$ of $\frac{30}{10} = \frac{7}{10}$ and $\frac{2}{10}$ over; $\frac{2}{10} = \frac{20}{100}$, $\frac{1}{4}$ of $\frac{20}{100} = \frac{5}{100}$, $\therefore \frac{3}{4} = \frac{7}{10} + \frac{5}{100} = \frac{75}{100}$ .... $\frac{75}{100}\big|\frac{15}{20}\big|\frac{3}{4}$

$\frac{5}{8} = \frac{1}{8}$ of $5 = \frac{1}{8}$ of $\frac{50}{10} = \frac{6}{10}$ and $\frac{2}{10}$ over; $\frac{2}{10} = \frac{20}{100}$, $\frac{1}{8}$ of $\frac{20}{100} = \frac{2}{100}$ and $\frac{4}{100}$ over; $\frac{4}{100} = \frac{40}{1000}$, $\frac{1}{8}$ of $\frac{40}{1000} = \frac{5}{1000}$, $\therefore \frac{5}{8} = \frac{6}{10} + \frac{2}{100} + \frac{5}{1000} = \frac{600+20+5}{1000}$ $= \frac{625}{1000}$ .... $\frac{625}{1000}\big|\frac{125}{200}\big|\frac{25}{40}\big|\frac{5}{8}$

$\frac{137}{512} = \frac{1}{512}$ of $137 = \frac{1}{512}$ of $\frac{1370}{10} = \frac{2}{10}$ and $\frac{346}{10}$ over; $\frac{346}{10} = \frac{3460}{100}$, $\frac{1}{512}$ of $\frac{3460}{100} = \frac{6}{100}$ and $\frac{388}{100}$ over; $\frac{388}{100} = \frac{3880}{1000}$, &c.

Mod. op.:

$$512)1370\left(\frac{2}{10} + \frac{6}{100} + \frac{7}{1000} + \frac{5}{10000} + \frac{7}{100000} + \frac{8}{1000000} + \frac{1}{10000000} + \frac{2}{100000000} + \frac{5}{1000000000}\right.$$

3460
3880
2960
4000
4160
640
1280
2560
....

Ans. $\frac{2}{10}+\frac{6}{100}+\frac{7}{1000}+\frac{5}{10000}+\frac{7}{100000}+\frac{8}{1000000}+\frac{1}{10000000}+\frac{2}{100000000}+\frac{5}{1000000000}$

$$=\frac{200000000+60000000+7000000+500000+70000+8000+100+20+5}{1000000000}$$

$=\frac{267578125}{1000000000}.$

Verification: $\frac{267578125}{1000000000}\Big|\frac{535156250}{2000000000}\Big|\frac{10703125}{40000000}\Big|\frac{2140625}{8000000}\Big|\frac{428125}{1600000}\Big|\frac{85625}{320000}\Big|\frac{17125}{64000}\Big|\frac{3425}{12800}\Big|\frac{685}{2560}\Big|\frac{137}{512}.$

Exercise II.

Reduce to decimal fractions and verify the results :

(1) $\frac{1}{5}$.	(4) $\frac{4}{5}$.	(7) $\frac{5}{8}$.	(10) $\frac{13}{20}$.	(13) $\frac{17}{64}$.
(2) $\frac{2}{5}$.	(5) $\frac{1}{8}$.	(8) $\frac{7}{8}$.	(11) $\frac{2}{25}$.	(14) $\frac{119}{128}$.
(3) $\frac{3}{5}$.	(6) $\frac{3}{8}$.	(9) $\frac{5}{16}$.	(12) $\frac{17}{40}$.	(15) $\frac{39}{125}$.

§ 3. In reducing $\frac{137}{512}$ to a decimal fraction, the denominator of any figure of the result may be known at once by knowing the place of that figure; thus the first figure has for denominator, 1 followed by one nought; the second, 1 followed by two noughts; the seventh, 1 followed by seven noughts, &c.; hence if the quotient be written 267578125, and if we know that the 2 is 2 tenths, the denominators of all the other figures will be known and may be omitted. To distinguish decimals from integers, a point is placed after the units' figure; thus 4·7 means 4 wholes and $\frac{7}{10}=4\frac{7}{10}$; similarly 123·456 means 123 units $+\frac{4}{10}+\frac{5}{100}+\frac{6}{1000}=123\frac{456}{1000}=\frac{123456}{1000}$. The absence of a numerator to any denominator must be indicated by a cipher; thus:

a. $\cdot42=\frac{4}{10}+\frac{2}{100}=\frac{42}{100}.$

b. $\cdot402=\frac{4}{10}+\frac{0}{100}+\frac{2}{1000}=\frac{402}{1000}.$

c. $\cdot042=\frac{0}{10}+\frac{4}{100}+\frac{2}{1000}=\frac{42}{1000}.$

d. $\cdot420=\frac{4}{10}+\frac{2}{100}+\frac{0}{1000}=\frac{420}{1000}=\frac{42}{100}.$

§ 4. Line *a* is of the same value as line *d*; line *c* is one-tenth of line *a*; from which it appears that a 0 added to either end of a decimal fraction produces the contrary effect to that which it produces on integers.

Integers.	Decimals.
0743 = 743	·7430 = ·743
7430 = 10 × 743	·0743 = $\frac{1}{10}$ of ·743

In words: In integers, a 0 on the left is of no effect; in decimals, a 0 on the right is of no effect. In integers, a 0 on the right *multiplies* by 10; in decimals, a 0 on the left *divides* by 10.

Read off: ·007. *Ans.* $\frac{7}{1000}$.

" ·04051. " $\frac{4}{100}+\frac{5}{10000}+\frac{1}{100000}=\frac{4051}{100000}$.

" 16·25. " 16 wholes $+\frac{2}{10}+\frac{5}{100}=16\frac{25}{100}=\frac{1625}{100}$.

Exercise III.

I. Give all the different readings of:

(1) ·1.	(6) 1·01.	(11) 3·15.
(2) ·01.	(7) ·11.	(12) 31·5.
(3) ·001.	(8) ·31.	(13) 128·053.
(4) ·000001.	(9) ·315.	(14) 12·8053.
(5) 1·1.	(10) ·0315.	(15) 1280·53.

II. Reduce to vulgar fractions at lowest terms:

(1) ·785.	(6) 6·03125.	(11) ·00055.
(2) ·1875.	(7) 603·125.	(12) ·505.
(3) ·73125.	(8) ·55.	(13) 5·05.
(4) ·603125.	(9) ·371.	(14) 8·888.
(5) ·128.	(10) ·00016.	(15) ·728.

§ 5. Addition.

Simplify ·68 + 43·159 + ·07 + 9·00124 + 453·69 + 8·9871 + 412.

Mod. op.:

```
   ·68
 43·159
   ·07
  9·00124
453·69
  8·9871
412
---------
927·58734
```

Arrange the addenda in column, plaçing units under units, tenths under tenths, &c. As the decimal point marks the units' place, the figures will fall into their proper places if the decimal points are under one another. Since the figures are arranged in the decimal scale, the same rules for carrying hold as in integers.

Add by vulgar fractions and by decimals, $2\frac{3}{5}$, $4\frac{5}{8}$, $7\frac{11}{20}$, $49\frac{13}{25}$.

By vulgar fractions:

	L.C.M.	200
2	40	120
4	25	125
7	10	110
49	8	104
62	200)	459(2
2		59

Ans. $64\frac{59}{200}$.

By decimals:

$2\frac{3}{5} = 2\cdot6$
$4\frac{5}{8} = 4\cdot625$
$7\frac{11}{20} = 7\cdot55$
$49\frac{13}{25} = 49\cdot52$

Ans. $64\cdot295$

$\frac{295}{1000} \mid \frac{59}{200}$.

Ans. By either method, $64\frac{59}{200}$.

Exercise IV.

Simplify by vulgar fractions and by decimals, shewing that the results agree :

(1) $7\frac{2}{5} + 4\frac{5}{8} + 9\frac{13}{20} + 11\frac{29}{32}$.

(2) $84\frac{13}{20} + 19\frac{11}{25} + 417\frac{19}{32} + 5043\frac{49}{64} + \frac{41}{50}$.

(3) $4\frac{27}{64} + 13\frac{17}{20} + 42\frac{37}{50} + 418\frac{19}{25} + 2\frac{13}{16} + 1\frac{1}{2}$.

(4) $5\frac{7}{8} + 13\frac{4}{5} + 19\frac{7}{16} + 7\frac{3}{20} + 18\frac{17}{40}$.

(5) $37\frac{5}{16} + 9\frac{4}{5} + \frac{2}{3}$ of $1\frac{4}{5} + \frac{7}{8}$ of $2\frac{2}{7} + \frac{3}{4}$ of $\frac{5}{8}$ of $\frac{7}{10}$.

(6) $9\frac{11}{32} + \frac{47}{100} + 11\frac{19}{128} + 3\frac{3}{5}$ of $4\frac{1}{9} + (\frac{2}{3} - \frac{1}{6})$.

§ 6. Subtraction.

From 17·08 take 9·643.

Mod. op.:

$$\begin{array}{r} 17\cdot08 \\ 9\cdot643 \\ \hline 7\cdot437 \end{array}$$

Subtract as in integers; where there is no figure in the minuend over one in the subtrahend, place or imagine a 0. (Cf. § 4.)

Exercise V.

Simplify by vulgar fractions and by decimals :

(1) $7\frac{2}{5} - 4\frac{5}{8}$.

(2) $84\frac{13}{20} - 17\frac{31}{32}$.

(3) $100\frac{2}{25} - 83\frac{17}{80}$.

(4) $100 - 17\frac{113}{625}$.

(5) $82\frac{1}{5} - 37\frac{11}{16}$.

(6) $5\frac{1}{2} - \frac{3}{4}$ of $1\frac{13}{24}$.

(7) $8\frac{1}{5} - 1\frac{1}{2}$ of $\frac{3}{16}$.

(8) $\frac{14}{25} - \frac{11}{64}$.

§ 7. Multiplication.

Case I. By a power of 10.

$$457{\cdot}6843 \times 10 = \frac{4576843}{10000} \times 10 = \frac{4576843}{1000} = 4576{\cdot}843.$$

Comparing the product with the multiplicand, we see that the figures are the same, the only difference being that the decimal point is one place further to the right, so that each figure is one place higher in the decimal scale, i.e. is ten times as valuable.

To multiply by 10, we have therefore only to shift the decimal point one place to the right; by 100, two places; by 1000, three places, and so on.

Rule: To multiply a decimal by any power of 10, shift the decimal point as many places to the *right* as there are ciphers in the multiplier, adding ciphers if necessary. Thus:

$$7{\cdot}63 \times 100000 = 763000.$$

Exercise VI.

By vulgar fractions and by decimals:

(1) $4\frac{7}{25} \times 10, 100, 1000, 10000.$

(2) $56\frac{19}{32} \times 10, 100, 1000, 10000.$

(3) $8\frac{11}{16} \times 10, 100, 1000.$

(4) $\frac{19}{64} \times 10, 100, 1000, 10000000.$

Case II. By any integer.

a. $457{\cdot}6843 \times 9 = \frac{4576843}{10000} \times 9 = \frac{41191587}{10000} = 4119{\cdot}1587.$

b. $74{\cdot}9375 \times 8 = \frac{749375}{10000} \times 8 = \frac{5995000}{10000} = 599{\cdot}5000 = 599{\cdot}5.$

c. ${\cdot}00731 \times 7000 = {\cdot}00731 \times 1000 \times 7 = 7{\cdot}31 \times 7 = 51{\cdot}17.$

In each case, the multiplicand being ten-thousandths, the product must also be ten-thousandths; in other words, the multiplicand having four decimal places, the product must have the same number of places. When the place of the decimal point is determined, ciphers to the right of the product may be rejected as superfluous.

Rule: To multiply a decimal by an integer, proceed as in common multiplication, and from the product mark off (counting from the right) as many decimal places as there are in the multiplicand.

Exercise VII.

By vulgar fractions and by decimals :

(1) $43\frac{27}{50} \times 7$; $435\frac{2}{5} \times 7$; $\frac{2177}{5000} \times 7$; $4\frac{177}{500} \times 7$.

(2) $47\frac{5}{8} \times 3, 7, 37, 4, 6, 8, 46, 468$.

(3) $\frac{315}{512} \times 5043, 64$.

By decimals only :

(4) $9000 \times 167{\cdot}432, {\cdot}00719, {\cdot}000001$.

(5) ${\cdot}0678 \times 512000$; ${\cdot}03625 \times 102400$.

Case III. By a decimal.

a. ${\cdot}43 \times 3{\cdot}784$.

$${\cdot}43 \times 3{\cdot}784 = \frac{43}{100} \times \frac{3784}{1000} = \frac{43 \times 3784}{100 \times 1000} = \frac{162712}{100000} = 1{\cdot}62712.$$

b. ${\cdot}03 \times {\cdot}0005$.

$${\cdot}03 \times {\cdot}0005 = \frac{3}{100} \times \frac{5}{10000} = \frac{15}{1000000} = {\cdot}000015.$$

c. $3{\cdot}175 \times 25{\cdot}6$.

$$3{\cdot}175 \times 25{\cdot}6 = \frac{3175}{1000} \times \frac{256}{10} = \frac{3175 \times 256}{1000 \times 10} = \frac{812800}{10000} = 81{\cdot}2800 = 81{\cdot}28.$$

From these three instances the following rule is obvious :

To multiply a decimal by a decimal, proceed as in common multiplication, and from the product mark off (counting from the right) as many decimal places as there are in the multiplier and multiplicand together. Any ciphers on the right of the product may be struck out *after* the position of the decimal point is determined. This rule evidently includes those given for Cases I. and II.

Exercise VIII.

(1) $16{\cdot}42 \times 4{\cdot}17$.

(2) $1{\cdot}642 \times 41{\cdot}7$.

(3) ${\cdot}1642 \times 417$.

(4) $164{\cdot}2 \times {\cdot}0417$.

(5) ${\cdot}3 \times {\cdot}4$; ${\cdot}03 \times {\cdot}004$; ${\cdot}03 \times 1$; ${\cdot}03 \times {\cdot}1$; ${\cdot}003 \times {\cdot}001$; ${\cdot}005 \times {\cdot}04$; ${\cdot}004 \times {\cdot}05$.

(6) $1{\cdot}1 \times {\cdot}011$; $1{\cdot}01 \times {\cdot}0101$; ${\cdot}04 \times {\cdot}04 \times {\cdot}05$; ${\cdot}1 \times {\cdot}1 \times {\cdot}01$; ${\cdot}7 \times {\cdot}4 \times {\cdot}3 \times 1000$.

(7) $72{\cdot}159 \times 3{\cdot}27$; $7{\cdot}2159 \times {\cdot}327$; ${\cdot}72159 \times {\cdot}0327$.

(8) $16{\cdot}875 \times 5{\cdot}12$; $1{\cdot}6875 \times 51{\cdot}2$; ${\cdot}16875 \times {\cdot}0512$.

By vulgar fractions and by decimals :

(9) $4\frac{5}{8} \times 1\frac{1}{2}$; $7\frac{4}{5} \times \frac{3}{16}$; $\frac{13}{64} \times \frac{1}{25}$.

(10) $4\frac{7}{25} \times \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{25}$.

(11) $4\frac{7}{25} \times \cdot 02, \cdot 03, \cdot 04, \cdot 05, \cdot 032, \cdot 302, 3\cdot 2, \cdot 00032$.

§ 8. Division.

Case I. By a power of 10.

$457\cdot 6843 \div 10$.

$$\frac{4576843}{10000} \div 10 = \frac{4576843}{100000} = 45\cdot 76843.$$

Comparing the quotient with the dividend, we see that the figures are the same, the only difference being that the decimal point is one place further to the left, so that each figure is one place lower in the decimal scale, i. e. it has one-tenth of the value.

To divide by 10, we have therefore only to shift the point one place to the left; by 100, two places; by 1000, three places, and so on.

Rule : To divide a decimal by any power of 10, shift the decimal point as many places to the *left* as there are ciphers in the divisor, prefixing ciphers if necessary. Thus,

$$7\cdot 63 \div 100000 = \cdot 0000763.$$

Exercise IX.

By vulgar fractions and by decimals :

(1) $32\frac{5}{16} \div 10, 100, 1000, 10000$.

(2) $4\frac{7}{25} \div 10, 100, 1000$.

(3) $56\frac{19}{32} \div 10, 100$.

Case II. By any integer.

a. $4379 \div 5$.

$$\begin{array}{r} 5\underline{)4379} \\ 875\cdot 8 \end{array}$$

Wording: 5 in 43, **8′**, carry 3; in 37, **7′**, carry 2; in 29, **5′**, carry 4 (units) = 40 (tenths); in 40, **8′** (tenths).

Verification :

$$\begin{array}{r} 875\cdot 8 \\ \underline{5} \\ 4379\cdot 0 \end{array}$$

b. $3101 \div 8$.

$$\begin{array}{r} 8\underline{)\,3101\quad} \\ 387{\cdot}625 \end{array}$$

Wording: 8 in 31, **3′**, carry 7; in 70, **8′**, carry 6; in 61, **7′**, carry 5; in 50, **6′**, carry 2; in 20, **2′**, carry 4; in 40, **5′**.

Verification:
$$\begin{array}{r} 387{\cdot}625 \\ \underline{8} \\ 3101{\cdot}\cancel{000} \end{array}$$

c. $123{\cdot}75 \div 4$.

$$\begin{array}{r} 4\underline{)\,123{\cdot}75\quad} \\ 30{\cdot}9375 \end{array}$$

Wording: 4 in 12, **3′**; in 3, **0′**, carry 3; in 37 (tenths), **·9′**, carry 1; in 15, **3′**, carry 3; in 30, **7′**, carry 2; in 20, **5′**.

Verification:
$$\begin{array}{r} 30{\cdot}9375 \\ \underline{4} \\ 123{\cdot}75\cancel{00} \end{array}$$

d. $4{\cdot}19 \div 800 = (4{\cdot}19 \div 100) \div 8 = {\cdot}0419 \div 8$.

$$\begin{array}{r} 8\underline{)\,{\cdot}0419\quad} \\ {\cdot}0052375 \end{array}$$

Wording: 8 in 0 (tenths), **·0′**; in 4, **0′**; in 41, **5′**, carry 1; in 19, **2′**, carry 3; in 30, **3′**, carry 6; in 60, **7′**, carry 4; in 40, **5′**.

Verification: ${\cdot}0052375 \times 800 = 4{\cdot}19\cancel{000}$.

e. ${\cdot}0143 \div 32$.

$$\begin{array}{r} 8\underline{)\,{\cdot}0143\quad} \\ 4\underline{)\,{\cdot}0017875} \\ {\cdot}000446875 \end{array}$$

Verification: ${\cdot}000446875 \times 32 = {\cdot}0143$.

f. $5{\cdot}643 \div 76$.

$$\begin{array}{l} 76)\,5{\cdot}643\,({\cdot}07425 \\ \quad\quad 323 \\ \quad\quad 190 \\ \quad\quad\; 380 \\ \quad\quad\; \ldots \end{array}$$

From these six instances the following rule is obvious: To divide a decimal by an integer proceed as in common division, placing the decimal point in the quotient as soon as the figure in the first place of decimals is "brought down," and adding ciphers to the successive remainders as required.

N.B. The examples given in this section must be "carried out" until there is no remainder.

EXERCISE X.

(Verify all these by multiplication.)

(1) $749\cdot682 \div 2, 4, 5, 8.$
(2) $32594\cdot73 \div 32, 128, 25, 1024.$
(3) $358677\cdot9 \div 99288.$
(4) $\cdot1 \div 64, 512, 50, 800.$
(5) $\cdot13 \div 52, 10400.$
(6) $\cdot5 \div 2, 4, 5, 8.$
(7) $\cdot01 \div 2, 4, 5, 8.$
(8) $\cdot0073 \div 16, 1600, 160000.$

By vulgar fractions and by decimals:

(9) $28\frac{11}{40} \div 2, 3, 4, 5, 8, 58.$
(10) $13 \div 80$; $429 \div 16000$; $8193 \div 163840000.$

CASE III. By a decimal.

By multiplying the dividend, the quotient is multiplied.

By multiplying the divisor, the quotient is divided.

By multiplying both dividend and divisor by the same number, the quotient remains unaltered. (But the remainder, if any, is still multiplied, being unaffected by the multiplication of the divisor.)

Hence to divide by a decimal, first of all multiply both divisor and dividend by such a power of 10 as will expel the decimal point from the divisor. This will reduce the problem to one under Case II.

$12\cdot307722 \div \cdot294 = 12307\cdot722 \div 294.$

```
294)12307·722(41·863
     547
     2537
      1852
        882
        ...
```

$\cdot005 \div 3\cdot2 = \cdot05 \div 32.$

```
8)·05
 ────────
4)·00625
 ────────
  ·0015625
```

Exercise XI.

By vulgar fractions and by decimals:

(1) $\cdot0073 \div \frac{4}{25}$.

(2) $\cdot1 \div \frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{4}{5}$.

(3) $93\frac{3}{5} \div \cdot2, \cdot3, \cdot4, \cdot5, \frac{4}{5}$.

(4) $417\cdot143 \div 12\frac{4}{5}, \frac{16}{125}, \frac{4}{3125}$.

(5) $\frac{4}{3125} \div 8\frac{24}{125}$.

(6) $1708\cdot4592 \div \cdot00024$.

(7) $6\frac{9}{16} \div 1\frac{1}{2}$.

(8) $6\frac{9}{16} \div 4\frac{3}{8}$.

(9) $28\frac{11}{40} \div \cdot58, \cdot058, 4\cdot875, \cdot4875, 48\cdot75$.

(10) $\cdot1 \div \cdot1$; $\cdot1 \div 1$; $1 \div \cdot1$; $\cdot01 \div \cdot0004$; $\cdot0004 \div \cdot01$.

§ 9. Find the value of ·06875 of a cwt.

1st method.

$$\cdot06875 = \frac{6875}{100000} = \frac{11}{160}\text{ cwt.} = \frac{11 \times \overset{7}{\cancel{112}}}{\underset{10}{\cancel{160}}}\text{ lbs.} = \frac{77}{10} = 7\frac{7}{10}\text{ lbs.}$$

Ans. $7\frac{7}{10}$ lbs.

2nd method.

$$\begin{array}{r l} \cdot06875 & \text{cwt.} \\ 4 & \\ \hline \cdot275\cancel{0}\cancel{0} & \text{qrs.} \\ 4 & \\ \hline 1\cdot1\cancel{0}\cancel{0} & \\ 7 & \\ \hline 7\cdot7 & \text{lbs.} \end{array}$$

Ans. $7\frac{7}{10}$ lbs.

Find the value of £·75003125.

1st method. $£\cdot75003125 = £\frac{75003125}{100000000} = £\frac{24001}{32000} = 15s.\ 0\frac{3}{400}d.$

2nd method.

$$\begin{array}{r l} £\cdot75003125 & \\ 20 & \\ \hline 15\cdot0006250\!\!\!/ & \text{shillings} \\ 12 & \\ \hline \cdot0075\cancel{0}\cancel{0} = \frac{75}{10000} = \frac{3}{400}d. & \end{array}$$

Ans. $15s.\ 0\frac{3}{400}d.$

Exercise XII.

(1) Find the value of £·721875.

(2) " ·6 of 1s.

(3) " ·045 of 1 cwt.

(4) Find the value of ·8 of a year.
(5) „ ·2345 of an hour.
(6) „ ·0109375 tons.
(7) „ ·06412 miles.

§ 10. Reduce $1\frac{3}{4}$ lbs. to the decimal of 1 cwt.

1st method. $1\frac{3}{4}$ lbs. $= \frac{7}{4}$ lbs. $= \dfrac{\overset{1}{\cancel{7}}}{4\times\underset{16}{\cancel{112}}}$ cwt. $= \frac{1}{64}$ cwt.

```
8)1·
 8)·125
   ·015625 cwt.
```

Ans. ·015625 cwt.

2nd method.

```
4)7
4)1·75 lbs.
 7)·4375
  4)·0625 qrs.
    ·015625 cwt.
```

Ans. ·015625 cwt.

Exercise XIII.

(1) Reduce 146 days to the decimal of 1 year.
(2) „ 77 lbs. „ 1 ton.
(3) „ 4 dwts., 15 grs. „ 1 oz. troy.
(4) „ 7 cwt., 3 qrs., $17\frac{1}{2}$ lbs. „ 1 cwt.
(5) „ 11*s*. $5\frac{1}{4}$*d*. „ £1.
(6) „ £4. 13*s*. $9\frac{3}{4}$*d*. „ £100.

CHAPTER III.

THE METRIC SYSTEM.

Previous to the French Revolution of 1789, France was divided into provinces, many of which had their own peculiar weights, measures, and even coinage. The Revolution introduced (March, 1795) a uniform system, which was, however, adopted slowly and

with difficulty. It only acquired universally legal force in 1840. It is now in use in its entirety in France and Belgium, and is partially adopted in the Zollverein, Italy, &c.

In this system, the distance from the North Pole to the Equator (the Quadrant) was divided into 10,000,000 equal parts, of which each was called a METRE (39·37079 inches). This length forms the unit of the whole system, and hence the name, "Metric System." Multiples and parts of this unit run decimally, i. e. by powers of 10; hence the system is a decimal system. These two features of the system, its choice of unit and its decimal character, are quite independent of one another, and each might exist without the other.

The Greek prefixes, *deca*, *hecto*, *kilo* and *myria*, are used to indicate decimal multiplication of the metre; and the Latin prefixes, *deci*, *centi*, *milli*, to indicate decimal subdivisions of the metre. Thus: LENGTH.

Myriametre	=	393707·9	Inches.
Kilometre	=	39370·79	
Hectometre	=	3937·079	
Decametre	=	393·7079	
METRE	=	39·37079	
Decimetre	=	3·937079	
Centimetre	=	·3937079	
Millimetre	=	·03937079	

SURFACE. The unit of Surface is a decametre square, and is called the ARE (= ·02471143 acres).

10000	square metres	= 1 hectare	= 2·471143	acres.
100	,, metres	= 1 ARE	= ·02471143	,,
1	,, metre	= 1 centiare	= ·0002471143	,,

SOLIDITY. The unit of Solidity is a cubic metre, and is called a STERE (= 35·32 cubic feet).

CAPACITY. The unit of Capacity is a cubic decimetre, and is called a LITRE (= ·22009687 imperial gallons).

Kilolitre	=	220·09687	Gallons.
Hectolitre	=	22·009687	
Decalitre	=	2·2009687	
LITRE	=	·22009687	
Decilitre	=	·022009687	
Centilitre	=	·0022009687	
Millilitre	=	·00022009687	

WEIGHT. The unit of weight is a cubic centimetre of distilled water at its maximum density (very nearly 40^{0} F.), and is called 1 GRAMME = ·00220606 lbs. av., 15·442 grains.

Ton or Millier	=2205·7	lbs. av. = ·9847 tons.
Quintal	=220·57	lbs. av.
Myriagramme	=22·057	lbs. av.
Kilogramme	=2·2057	lbs. av.
Hectogramme	= 1544	grains.
Decagramme	= 154·4	grains.
GRAMME...............	= 15·44	grains.
Decigramme	= 1·544	grains.
Centigramme............	= ·1544	grains.
Milligramme	= ·01544	grains.

MONEY. 5 grammes of silver of a certain fixed fineness are coined into 1 FRANC = 10 decimes = 100 centimes = $9\frac{1}{2}$ pence* nearly. (In familiar language, the decime is called *deux sous*).

Hence 7·485 metres = 7 metres, 4 decimetres, 8 centimetres and 5 millimetres; or 7 metres, 485 millimetres, &c.

14·89 litres = 14 litres, 8 decilitres, 9 centilitres; or 14 litres, 89 centilitres, &c., and so on.

Find the cost of 10, 100, 1000, 10,000 metres, at fr. 2·35 each.

1 metre	costs fr.	2·35
10	,,	23·5
100	,,	235
1000	,,	2350
10000	,,	23500

Find the cost of 1 kilogramme at fr. 23·5 per quintal.

1 quintal costs fr. 23·5
1 kilogramme ,, ·235 = $23\frac{1}{2}$ centimes.

Find the cost of 385·75 metres at fr. 4·65 per metre.

```
    38575
      465
   ------
   192875
  231450
 154300
 --------
 1793·7375
```

Ans. fr. 1793·73.

N.B. Fractions of a centime are disregarded.

* £1 sterling at par = fr. 25·2215.

Find the cost of 273·42 litres at fr. 123·5 per hectolitre.

1 hectolitre costs fr. 123·5, ∴ 1 litre costs fr. 1·235.

```
    27342
     1235
   ------
   136710
   82026
  54684
 27342
 ---------
 337·67370          Ans. fr. 337·67.
```

Find the cost in English money of 17 litres, if 1 hectolitre costs fr. 454·5, and £1 = fr. 25·25.

£ x	17 litres.
litres 100	454·5 fr.
fr. 25·25	£1.

```
  ·06                         4545
   20                       --------
 -----               2525) 7726·5 (3·06
  1·2                      151 50
   12                       . . . .
 -----
 2 4/10 = 2/5              Ans. £3·06 = £3. 1s. 2 2/5 d.
```

Find the cost per litre, if 468·59 litres cost fr. 1274.

```
46859)127400(2·718
      336820
        88070
        412110
         372380          Ans. fr. 2·72 nearly.
```

Exercise XIV.

(1) Find the sum of 4·173, ·0089, ·2375, ·1, ·01, 246.

(2) What number exceeds ·999 by ·001?

(3) From what vulgar fraction must ·625 be subtracted to leave ·295?

(4) By vulgar fractions and by decimals, find $4\frac{7}{8}$ + ·01375.

(5) ·0876 exceeds a certain quantity by ·00876. Find it.

(6) There are two numbers; the greater is 3·142857; their difference is ·001267. Find the less.

(7) If the year is reckoned at $365\frac{1}{4}$ days instead of 365·242264 days, what will be the amount of error in 19 centuries?

(8) Express ·4984 of a day in hours, minutes and seconds.

(9) What fraction contains ·125 ·486 times?

(10) Of what number is ·4 the 25th part?

(11) How many times can ·0085 be subtracted from ·18, and what will be over?

(12) How many times can ·029 be taken out of ·3786, and what will be over?

(13) How much must be subtracted from ·710267875 to leave the largest multiple of ·000000046275 it contains?

(14) Simplify $(7·13 + 3·875) + (7·13 - 3·875) + (7·13 \times 3·875) + (7·13 \div 3·875)$.

(15) Find the reciprocal of ·64.

(16) $72·315 \times 1000$, $·001$, $\frac{1}{100}$, $\frac{1}{·001}$.

(17) $72·315 \div 1000$, $·001$, $\frac{1}{100}$, $\frac{1}{·001}$.

(18) Find the cost of 12·5 kilolitres at 3·75 francs per litre.

(19) Find the cost per metre if 437·75 metres cost 1805·71875 francs.

(20) Find the value of £·021875 + ·375*s.* + 4·75*d.*

(21) Find the value of £·08 + ·08*s.* + ·09*d.*, and express the result as a decimal of £1.

(22) Add together £1·3625, ·75 of 13*s.* 4*d.*, and $\frac{2}{25}$ of £20.

(23) Reduce ·06 of ·42 of a guinea to the decimal of £1.

(24) Divide ·010875 by ·00625, and verify by vulgar fractions.

(25) Reduce to a vulgar fraction $·7 + \frac{3}{11}$ of $·825 + 4·13$.

(26) Simplify:

a. $\dfrac{1·4 \times ·035}{·00014}$.

b. $\dfrac{1·4 + ·035}{·00014}$.

c. $\dfrac{1·4 - ·035}{·00014}$.

d. $\dfrac{·1 \times ·01 \times ·001 \times ·001}{·001 \times ·0001}$.

e. $\dfrac{·13 \times ·14 \times ·01 + ·12 \times ·13 \times ·01 - ·12 \times ·14 \times ·02}{·01 \times ·2 \times ·01}$.

f. $\dfrac{1·85}{29·6} + \dfrac{·51}{·425}$ of $\dfrac{6·875}{22} + ·0625$.

(27) Reduce to metres :

17·35 myriametres,	17·35 decimetres,
17·35 kilometres,	17·35 centimetres,
17·35 hectometres,	17·35 millimetres.

(28) How many square decimetres in a square metre, and how many cubic centimetres in a cubic metre?

(29) Express 500 cubic metres as cubic yards, taking 1 metre = 39·37 inches.

(30) Find G.C.M. of 64·09 and 7·395.

(31) A sum of money is divided among three persons; the first receives ·375 of the whole, the second ·6, and the third £2·125. Find it.

(32) A does ·375 of a piece of work in 2·25 days, and B does the remainder of it in 3·75 days. How many such pieces of work would A and B together do in 12 days?

(33) Find the value of the following: ·175 tons + ·195 cwt. + ·145 qr. + ·15 lbs. + ·2 oz.

(34) If a gramme is 15·442 grains, and a metre 39·37 inches, how many grammes are there in 1000 grains, and how many metres in a mile?

(35) Simplify $\dfrac{\cdot 002 \times 1\cdot 75 \div \cdot 00007}{1\frac{2}{3} \div \frac{1}{3}}$.

(36) Find the sum, difference, product and quotient (the greater being divided by the less) of ·016 and ·02235.

(37) Which is more, and by how much, £·4999 or £·5?

(38) Find the cost of 1 millier at fr. 1·75 per kilog.

(39) Find the cost in English money per lb. av., if 950 grammes cost fr. 21·23, and fr. 1 = $9\frac{1}{2}d$.

(40) Express in English units: $7\frac{1}{2}$ quintals, 580 kiloms., 85·6 ares, $1437\frac{1}{2}$ francs, 570 litres.

(41) Reckoning £1 = fr. 25·2215, find the value in French money of £450.

CHAPTER IV.

RECURRING DECIMALS.

§ 1. Reduce $\frac{1}{3}$ to a decimal.

$$3\overline{)\,1\cdot\quad}$$
$$\cdot333,\ \&c.$$

We see that $\frac{1}{3}$ cannot be accurately expressed as a decimal, and the question arises, Are there many such fractions? If by trial we classify all the fractions from $\frac{1}{2}$ to $\frac{1}{16}$, we shall find that

$\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}$, are reducible;
$\frac{1}{3}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}$, are not reducible;

and if the trial is carried further, we shall find that the preponderance of non-reducible fractions continually increases. Either, then, decimal fractions are comparatively useless, being only applicable to a very few fractions, or methods must be found for manipulating non-reducible fractions.

In the reduction of $\frac{1}{3}$, we find that the quotient 3 continually recurs, and this is expressed thus, $\cdot\dot{3}$; similarly, ·575757, &c., is written $\cdot\dot{5}\dot{7}$, and ·41374137, &c., is written $\cdot\dot{4}13\dot{7}$. Sometimes only a part of the quotient will recur, as in ·6741094109, &c., which is written $\cdot67\dot{4}10\dot{9}$, the dots being placed over the first and last figures of the recurring "period." Decimal fractions where ALL the figures recur are called PURE CIRCULATORS; those where some of the figures do not recur are called MIXED CIRCULATORS.

§ 2. In dividing 1 by 3, we obtained for quotient $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$ ad infinitum, with the successive remainders, $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$, &c., also ad infinitum. Wherever we stop, we disregard this remainder, and therefore obtain an inaccurate result. This inaccuracy, however, must continually diminish, as will appear from the following table:

·3	is less than	·33,	which is less than	$\frac{1}{3}$,
·33	,,	·333,	,,	$\frac{1}{3}$,
·333	,,	·3333,	,,	$\frac{1}{3}$,

and so on, ad infinitum. The fractions ·3, ·33, ·333, ·3333, &c., continually increase, and yet fall short of $\frac{1}{3}$; therefore the more figures we take, the nearer we approach to $\frac{1}{3}$; but as there is always a remainder, we never actually reach it.

Reduce $\frac{2}{7}$ to a decimal.

$$7\underline{)2}$$
$$\cdot\dot{2}8571\dot{4}$$

$\frac{2}{7} > \cdot 2$ by $\frac{3}{35}$ .... A

$\frac{2}{7} > \cdot 28$ by $\frac{1}{175}$, which reduces the error by .... $\frac{2}{25}$ ... B

$\frac{2}{7} > \cdot 285$ by $\frac{1}{1400}$, which further reduces the error by $\frac{1}{200}$ ... C

$\frac{2}{7} > \cdot 2857$ by $\frac{1}{70000}$, " " $\frac{7}{10000}$ ... D

$\frac{2}{7} > \cdot 28571$ by $\frac{3}{700000}$, " " $\frac{1}{100000}$ ... E

$\frac{2}{7} > \cdot 285714$ by $\frac{1}{3500000}$, " " $\frac{1}{250000}$ ... F

&c.

Estimate of error in different Units.

	£. s. d.	lb. av.	Cwt.	Ton.	Yd.	Mile.	7920 miles, Earth's diameter.	93500000 miles, Sun's distance.
A	1s. $8\frac{4}{7}$d.	$1\frac{13}{35}$ oz.	$9\frac{3}{5}$ lb.	1 cwt. 2 qrs. 24 lb.	$3\frac{3}{35}$ in.	$150\frac{6}{7}$ yd.	$678\frac{6}{7}$ m.	$8014285\frac{5}{7}$ miles.
B	$1\frac{13}{35}$d.	$1\frac{81}{175}$ dr.	$10\frac{6}{25}$ oz.	$12\frac{4}{5}$ lb.	$\frac{36}{175}$ in.	$10\frac{2}{35}$ yd.	$45\frac{9}{35}$ m.	$534285\frac{5}{7}$ "
C	$\frac{24}{35}$ f.	5 gr.	$1\frac{7}{25}$ oz.	$1\frac{3}{5}$ lb.	$\frac{9}{350}$ in.	$1\frac{9}{35}$ yd.	$5\frac{23}{35}$ m.	$66785\frac{5}{7}$ "
D	$\frac{12}{875}$ f.	$\frac{1}{10}$ gr.	$11\frac{1}{5}$ gr.	$8\frac{24}{125}$ dr.	$\frac{9}{17500}$ in.	$\frac{792}{875}$ in.	$199\frac{23}{175}$ yd.	$1335\frac{5}{7}$ "
E	$\frac{18}{4375}$ f.	$\frac{3}{100}$ gr.	$3\frac{9}{25}$ gr.	$2\frac{286}{625}$ dr.	$\frac{27}{175000}$ in.	$\frac{1188}{4375}$ in.	$59\frac{647}{875}$ yd.	$400\frac{5}{7}$ "
F	$\frac{6}{21875}$ f.	$\frac{1}{500}$ gr.	$\frac{28}{125}$ gr.	$4\frac{12}{25}$ gr.	$\frac{9}{875000}$ in.	$\frac{396}{21875}$ in.	$3\frac{4299}{4375}$ yd.	$26\frac{5}{7}$ "
		&c.			&c.		&c.	

This table may be read thus: $\frac{2}{7}$ of a unit exceeds ·2 of it by $\frac{3}{35}$ of it. If this unit be £1, this error = 1s. $8\frac{4}{7}$d.; if 1 lb., the error = $1\frac{13}{35}$ oz., &c. In line B, this error is only $\frac{1}{175}$ of £1 or of 1 lb., &c.

EXERCISE XV.

(1) Construct a similar table with $\frac{7}{11}$ and with $\frac{5}{13}$.

(2) Estimate the difference between :

a. $\frac{5}{17}$ of 1 mile and ·2941 of 1 mile.

b. £$\frac{13}{19}$ and £·6842.

Examining this table, we find (*a*) that the error continually diminishes; (*b*) that the larger the unit handled, the further must the process be carried to render the error insignificant. Thus the lb. column shews a microscopic error in the 4th line, whilst with the tons the error in this line is still appreciable. The same fact is yet more strikingly apparent if we compare the yard column with that of the sun's distance.

The continually diminishing error can be made as small as we please; that is to say, it can be made less than any assigned quantity, however small. Thus,

$\frac{2}{7}$ > ·2 and < ·3 ; it lies between ·2 and ·3, and does not differ from either by ·1.
$\frac{2}{7}$ > ·28 and < ·29 ; „ ·28 and ·29, „ ·01.
$\frac{2}{7}$ > ·285 and < ·286 ; „ ·285 and ·286, „ ·001.
⋮
$\frac{2}{7}$ > ·285714 and < ·285715 ; „ ·285714 and ·285715, „ ·000001.

Suppose a problem is given to which the answer must not be wrong by so much as $\frac{1}{24730}$ of the unit; $\frac{1}{24730}$ > ·00001. If the answer be a decimal correct to 5 places, its error < ·00001, and the required degree of accuracy is attained.

The series

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\cdots\cdots$$
$$\frac{285714}{1000000}+\frac{285714}{1000000000000}+\frac{285714}{1000000000000000000}+\cdots\cdots$$

respectively approach $\frac{1}{3}$ and $\frac{2}{7}$, to which, by taking a sufficient number of decimal places, they may be made as near as we please. $\frac{1}{3}$ and $\frac{2}{7}$ are called the LIMITS of these series.

"The word limit implies a fixed magnitude, to which another and a variable magnitude may be made as nearly equal as we please, it being impossible, however, that the variable magnitude can absolutely attain or be equal to the fixed magnitude. In this strict sense of the word, there are two conditions which must be fulfilled before A can be called the limit of P : first, P must never become equal to

A; secondly, P must be capable of being made as nearly equal to A as we please."*

Since this limit can never be reached, we must stop somewhere near it, and we have to settle the principle on which the stoppage is to be regulated. If the distance between 0 and 1 be divided decimally, we shall find the following stations marked out along the line:

0 ·1 ·2 ·3 ·4 ·5 ·6 ·7 ·8 ·9 1 A

Again: the distance between, say ·1 and ·2, may be subdivided, which, under a glass magnifying it ten-fold, would look thus:

·1 ·11 ·12 ·13 ·14 ·15 ·16 ·17 ·18 ·19 ·2 B

Similarly, any one of these intervals, say ·13 and ·14, might be further magnified ten-fold, and would look thus:

·13 ·131 ·132 ·133 ·134 ·135 ·136 ·137 ·138 ·139 ·14 C

Now take $\frac{2}{7} = \dot{2}8571\dot{4}$

0 ·1 ·2 ·3 ·4 ·5 ·6 ·7 ·8 ·9 1 D

$\frac{1}{7}$ $\frac{2}{7}$ $\frac{3}{7}$ $\frac{4}{7}$ $\frac{5}{7}$ $\frac{6}{7}$

·2——·3 magnified:

·2 ·21 ·22 ·23 ·24 ·25 ·26 ·27 ·28 ·29 ·3 E

$\frac{2}{7}$

·28——·29 magnified:

·28 ·281 ·282 ·283 ·284 ·285 ·286 ·287 ·288 ·289 ·29 F

$\frac{2}{7}$

·285——·286 magnified:

·285 ·2851 ·2852 ·2853 ·2854 ·2855 ·2856 ·2857 ·2858 ·2859 ·286 G

$\frac{2}{7}$

·2857——·2858 magnified:

·2857 ·-571 ·-572 ·-573 ·-574 ·-575 ·-576 ·-577 ·-578 ·-579 ·2858 H

$\frac{2}{7}$

And so on. This magnifying of successively smaller intervals, which we may imagine carried on ad infinitum, brings home to us the idea of the infinite divisibility of a line; for, however small the

* Penny Cycl., art. LIMIT.

intervals are made, or, in other words, however numerous the points to which names are given, there must always be between them an infinitely larger number of points left unnamed. Professor De Morgan, in one of his lectures, has compared Arithmetic to the piano and Geometry to the violin: the former instrument by its structure can only give name to lengths of string at certain intervals; the latter omits no point along the whole line.

We see now that *midway* between	0	and 1	lies ·5
„	·1	·2	·15
„	·13	·14	·135
„	·2	·3	·25
„	·28	·29	·285
„	·285	·286	·2855
„	·2857	·2858	·28575, &c.

Line A shews that all quantities below ·5 are nearer to 0 than to 1; ·5 is midway, and those above ·5 are nearer to 1. $\frac{2}{7}$ is nearer to ·3 than to ·2, being above ·25; $\frac{2}{7}$ is nearer to ·29 than to ·28, being above ·285; $\frac{2}{7}$ is nearer to ·286 than to ·285, being above ·2855; $\frac{2}{7}$ is nearer to ·2857 than to ·2858, being below ·28575, and so on. If we wish to express $\frac{2}{7}$ as a decimal to

one place, we shall be nearest the truth by calling it			·3
two places,	„	„	·29
three	„	„	·286
four	„	„	·2857
five	„	„	·28571
six	„	„	·285714
seven	„	„	·2857143, &c.

Rule: *In* CURTAILING *a decimal, if the first figure rejected be less than* 5, *make no change in the last figure retained; if more than* 4, *increase by* 1 *the last figure retained.*

If the first figure rejected be 5, it is immaterial whether we adopt the higher or the lower value if the fraction terminates at the 5; but as there generally are more figures after the 5, the higher value is somewhat more correct.

EXERCISE XVI.

(1) Curtail to 5 places $\cdot430718$, $\cdot010\dot{2}0\dot{3}$, $\cdot430\dot{7}9\dot{8}$.
(2) „ to 4 places $\cdot\dot{6}$, $\cdot4$, $\cdot\dot{2}\dot{7}$, $\cdot2\dot{7}$, $4\cdot\dot{0}\dot{9}$, $4\cdot0\dot{9}$.
(3) „ to 7 places $\cdot\dot{9}$, $\cdot\dot{0}\dot{9}$, $\cdot000\dot{9}$, $\cdot37\dot{9}$, $\cdot42\dot{9}$, $\cdot35\dot{9}$.
(4) „ $999\cdot\dot{9}$ to nearest integer.
(5) Reduce $\frac{1}{17}$ to a decimal to 1, 2, 3......16 places.

§ 3. ADDITION AND SUBTRACTION.

Work the following by vulgar fractions and by decimals, correct to 4 places:

$$7\tfrac{2}{3} + 13\tfrac{3}{7} + 5\tfrac{5}{14} + 43\tfrac{7}{15} + \tfrac{11}{18} + 6\tfrac{4}{9} + 10\tfrac{6}{35} + 100\tfrac{8}{13} + 3\tfrac{9}{91} + 1\tfrac{12}{65}.$$

	8190	
7	2730	5460
13	1170	3510
5	585	2925
43	546	3822
	455	5005
6	910	3640
10	234	1404
100	630	5040
3	90	810
1	126	1512
192	8190)	33128 (4

$$\frac{368}{8190} = \frac{184}{4095}$$

4095) 184·00 (·04493
20200
38200
13450
&c.

$7\frac{2}{3}$ =	7·66667	$\cancel{6}$
$13\frac{3}{7}$ =	13·42857	$\cancel{1}$
$5\frac{5}{14}$ =	5·35714	$\cancel{2}$
$43\frac{7}{15}$ =	43·46667	$\cancel{6}$
$\frac{11}{18}$ =	·61111	$\cancel{1}$
$6\frac{4}{9}$ =	6·44444	$\cancel{4}$
$10\frac{6}{35}$ =	10·17143	$\cancel{8}$
$100\frac{8}{13}$ =	100·61538	$\cancel{4}$
$3\frac{9}{91}$ =	3·09890	$\cancel{1}$
$1\frac{12}{65}$ =	1·18462	$\cancel{5}$
	192·04493	

Ans. 192·0449.

N.B. To insure the correctness of the last place required (here the fourth), it is well to carry on the fractions *to one place more* than is specified.

Exercise XVII.

Simplify the following by vulgar fractions and by decimals :

(1) $\frac{13}{18} + 6\frac{8}{15} + 9\frac{11}{20} + 100\frac{13}{30}$ to 5 places.

(2) $10\frac{4}{11} + 9\frac{5}{13} + 6\frac{2}{3} + 8\frac{112}{143} + 3\frac{5}{9}$ to 5 places.

(3) $8\frac{2}{3} + 9\frac{5}{7} + 3\frac{11}{21} + 6\frac{23}{77} + 4\frac{5}{6} + 10\frac{13}{33} + 2\frac{7}{9}$ to 5 places.

(4) $8\frac{1}{2} + \frac{5}{6} + \frac{11}{14} + 7\frac{3}{10} + 14\frac{17}{21} + \frac{11}{15} + \frac{12}{35}$ to 5 places.

(5) $8\frac{3}{5} + 7\frac{5}{8} + 2\frac{4}{7} + 9\frac{13}{20} + 11\frac{4}{35} + 10\frac{15}{56} + 12\frac{33}{70}$ to 5 places.

(6) $\frac{2}{7}$ of $18 + \frac{3}{5}$ of $1\frac{4}{21}$ to 4 places.

(7) $\frac{13}{18} - \frac{8}{21}$ to 5 places.

(8) $13\frac{2}{75} - 3\frac{8}{15}$ to 5 places.

(9) $8\frac{4}{15} - 7\frac{11}{25}$ to 5 places.

(10) $7\frac{4}{7} - \frac{1}{2}$ of $8\frac{3}{11}$ to 5 places.

(11) $\frac{2}{3}$ of $6\frac{1}{2} - \frac{2}{5}$ of 4 to 5 places.

(12) $5\frac{1}{3}$ of $4\frac{1}{2} - 3\frac{1}{4}$ of $3\frac{1}{5}$ to 5 places.

(13) $\frac{1}{3} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17}$ to 5 places.

By decimals only :

(14) $\frac{5}{19} + \frac{8}{17} + \frac{4}{23} + \frac{11}{29}$ to 7 places.

(15) $\frac{14}{17} - \frac{7}{12}$ to 7 places.

(16) $(\frac{31}{41} + \frac{14}{43}) - (\frac{31}{41} - \frac{14}{43})$ to 7 places.

(17) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$ to 4 places.

(18) $(\frac{17}{59} + \frac{14}{61}) + (\frac{17}{59} - \frac{14}{61})$ to 4 places.

(19) $\cdot 4\dot{1}\dot{7} + 4\cdot 3\dot{1}6\dot{2} + 71\cdot\dot{5}\dot{8} + 4\cdot 3\dot{4}8\dot{7}$ to 3 places.

(20) $\cdot\dot{8}246\dot{1} + 43\cdot\dot{7}86\dot{2} - 17\cdot\dot{1}76\dot{4}$ to 5 places.

(21) $\frac{1}{3} + \dfrac{1}{3\times 3} + \dfrac{1}{3\times 3\times 3} + \dfrac{1}{3\times 3\times 3\times 3} + \ldots\ldots$ to 5 places.

(22) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\ldots$ to 6 places.

(23) $1 + 1 + \frac{1}{2} + \dfrac{1}{2\times 3} + \dfrac{1}{2\times 3\times 4} + \dfrac{1}{2\times 3\times 4\times 5} + \ldots\ldots$ to 9 places.

(24) $16 \times (\frac{1}{5} - \dfrac{1}{3\times 5^3} + \dfrac{1}{5\times 5^5} - \dfrac{1}{7\times 5^7} + \dfrac{1}{9\times 5^9} - \dfrac{1}{11\times 5^{11}} + \&c.) - \frac{4}{239}$ to 5 places.

§ 4. MULTIPLICATION.

CASE I. By a power of 10.

By vulgar fractions and by decimals, correct to 6 places, multiply $4\frac{5}{7}$ by 10, 100 and 10000.

$4\frac{5}{7}\times 10=47\frac{1}{7}$	$4\frac{5}{7}=4\cdot 714285714......$
$=47\cdot 142857$	$4\frac{5}{7}\times 10=47\cdot 142857$
$4\frac{5}{7}\times 100=471\frac{3}{7}$	$4\frac{5}{7}=4\cdot 714285714......$
$=471\cdot 428571$	$4\frac{5}{7}\times 100=471\cdot 428571$
$4\frac{5}{7}\times 10000=47142\frac{6}{7}$	$4\frac{5}{7}=4\cdot 71428571428\,.....$
$=47142\cdot 857143'$	$4\frac{5}{7}\times 10000=47142\cdot 857143'$

EXERCISE XVIII.

(1) $\dot{3}\times 10$, 1000, 100000, to 3 places.

(2) $4\cdot\dot{7}\dot{2}\times 10$, 100, 1000000, to 4 places.

By vulgar fractions and by decimals to 6 places:

(3) $5\frac{8}{9}\times 10$, 100, 10000.

(4) $\frac{7}{12}\times 10$, 100, 10000.

(5) $\frac{13}{14}\times 10000$.

(6) $\frac{5}{17}\times 100$.

(7) $\frac{1}{7200}\times 1000000$.

CASE II. By any integer.

By vulgar fractions and by decimals, correct to 4 places, find $19\frac{5}{11}\times 7$.

By vulgar fractions:

$$19\tfrac{5}{11}\times 7=136\tfrac{2}{11}=136\cdot 1818.$$

By decimals:

$$\begin{array}{r|l}
19\tfrac{5}{11}\times 7=19\cdot\dot{4}\dot{5}\times 7=19\cdot 4545 & 4 \\
19\cdot 4545 & 4 \\
19\cdot 4545 & 4 \\
19\cdot 4545 & 4 \\
19\cdot 4545 & 4 \\
19\cdot 4545 & 4 \\
19\cdot 4545 & 4 \\
\hline
136\cdot 1817 & 8 \\
8 &
\end{array}$$

Ans. 136·1818.

If we had not multiplied the 4's to the right of the vertical line, our fourth place would have been greatly inaccurate, as we should

have lost the carriage. Even with this multiplication, the fourth place is inaccurate, unless we allow for the rejection of the high figure in the fifth place. We therefore proceed as follows :

$$\begin{array}{r} 19{\cdot}45454 \\ 7 \\ \hline 136{\cdot}1818 \end{array}$$

Wording: 28, carry *three;* 35, 38' carry 3, &c.

It is most important to remember that when we multiply the first figure rejected, we carry the "nearest ten." Thus to 21, 22, 23 and 24, 20 is the nearest ten; to 26, 27, 28 and 29, 30 is the nearest ten; for 25, it is better to carry 3 than 2, as figures following the first rejected might increase, but could not decrease, the 25. In other words, if the units' figure be above 4, carry the higher ten; if below 5, the lower.

Multiply $\frac{11}{19}$ by 600 to 5 places.

By vulgar fractions :

$$\tfrac{11}{19} \times 600 = \tfrac{6600}{19} = 347\tfrac{7}{19} = 347{\cdot}368421.$$

By decimals :

$$\tfrac{11}{19} \times 600 = {\cdot}5789473684 \times 600 = 57{\cdot}89473684 \times 6.$$

$$\begin{array}{r} 57{\cdot}89473684 \\ 6 \\ \hline 347{\cdot}36842 \end{array}$$

Hence, to multiply by any multiple of a power of 10, first multiply by the power of 10 by shifting the point. It will be observed that, in preparing the multiplicand, we must carry out the original decimal one figure more for every cipher. In the simplest form the sum would stand thus :

$$\tfrac{11}{19} \times 600 = {\cdot}57894737 \times 600.$$

$$\begin{array}{r} 57{\cdot}894737 \\ 6 \\ \hline 347{\cdot}36842 \end{array}$$

Exercise XIX.

By vulgar fractions and by decimals :

(1) $\frac{8}{13} \times 7000$ to 4 places.
(2) $\frac{5}{21} \times 300$,,
(3) $\frac{7}{11} \times 80000$,,
(4) $\frac{4}{19} \times 19000$,,

(5) $\frac{5}{18} \times 90000$ to 4 places.
(6) $53\frac{1}{120} \times 50$ „
(7) $\frac{30}{37} \times 2000$ „
(8) $\frac{41}{42} \times 90000$ „
(9) $3\frac{3}{11} \times 6000000$ „

By decimals only:

(10) $3{\cdot}4\dot{3}\dot{1} \times 500$ to 4 places.
(11) $416{\cdot}7\dot{1} \times 4000$ „
(12) $\cdot\dot{7}8\dot{4} \times 30000$ „

Multiply $\frac{45}{73}$ by 427 to 6 places.

By vulgar fractions:

$$\frac{45}{73} \times 427 = \frac{19215}{73} = 263{\cdot}219178\cancel{0}.$$

By decimals:

$$\frac{45}{73} \times 427 = {\cdot}616438356164 \ldots\ldots \times 427.$$

$$\text{For } 400 \left\{ \begin{array}{r} {\cdot}616438356\cancel{164} \\ 4 \\ \hline 246{\cdot}575342 \end{array} \right.$$

$$\text{For } 20 \left\{ \begin{array}{r} {\cdot}616438356 \\ 2 \\ \hline 12{\cdot}328767 \end{array} \right. \qquad \begin{array}{r} 246{\cdot}575342 \\ 12{\cdot}328767 \\ 4{\cdot}315068 \\ \hline 263{\cdot}219177 \end{array}$$

$$\text{For } 7 \left\{ \begin{array}{r} {\cdot}616438356 \\ 7 \\ \hline 4{\cdot}315068 \end{array} \right.$$

This process may be contracted thus:

$$\begin{array}{r} {\cdot}616438356 \\ 724 \\ \hline 246{\cdot}575342 \\ 12{\cdot}328767 \\ 4{\cdot}315068 \\ \hline 263{\cdot}219177 \end{array}$$

Notice that the figures of the multiplier are reversed, each figure of the multiplier being written beneath that decimal place of the multiplicand which will give the last decimal place required. It will also be seen that the last figure was not accurate, being 7 instead of 8. Had the question been worked to seven places, the sixth would have been accurate.

EXERCISE XX.

By vulgar fractions and by decimals :

(1) $43\frac{40}{41} \times 259$ to 4 places.

(2) $\frac{16}{29} \times 1043$ „

(3) $1\frac{4}{11} \times 3020$ „

(4) $\frac{14}{15} \times 3015$ to 4 places.

(5) $10\frac{11}{19} \times 9500$ „

(6) $13\frac{143}{2048} \times 8432$ „

CASE III. By a fraction.

Multiply $\frac{4}{23}$ by $\frac{2}{5}$ to 4 places.

By vulgar fractions :

$$\frac{4}{23} \times \frac{2}{5} = \frac{8}{115} = \cdot 069\cancel{5}\cancel{6}. \atop \quad 6$$

Ans. ·0696.

By decimals :

$$\frac{4}{23} \times \frac{2}{5} = \cdot 173913\ldots\ldots \times \cdot 4 = \cdot 0173913 \times 4.$$

$$\begin{array}{r} \cdot 01739 \\ 4 \\ \hline \cdot 0696 \end{array}$$

Ans. ·0696.

Multiply $\frac{14}{15}$ by $\frac{1}{125}$ to 6 places.

By vulgar fractions :

$$\frac{14}{15} \times \frac{1}{125} = \frac{14}{1875} = \cdot 0074\dot{6}.$$

Ans. ·007467.

By decimals :

$$\frac{14}{15} \times \frac{1}{125} = \cdot 9\dot{3} \times \cdot 008 = \cdot 0009\dot{3} \times 8.$$

$$\begin{array}{r} \cdot 0009333 \\ 8 \\ \hline \cdot 007466 \end{array}$$

Ans. ·007466.

Multiply $542\frac{13}{88}$ by $314\frac{15}{16}$ to 4 places.

By vulgar fractions :

$$542\tfrac{13}{88} \times 314\tfrac{15}{16} = \frac{47709}{88} \times \frac{5039}{16} = \frac{240405651}{1408} = 170742\tfrac{915}{1408} = 170742 \cdot 649\cancel{8}\cancel{5}. \atop \qquad 9$$

Ans. 170742·6499.

By decimals :

$$542\tfrac{13}{88} \times 314\tfrac{15}{16} = 542 \cdot 14\dot{7}\dot{2} \times 314 \cdot 9375.$$

$$\text{For } 314 \left\{ \begin{array}{r} 542 \cdot 1477272 \\ 413 \\ \hline 162644 \cdot 3182 \\ 5421 \cdot 4773 \\ 2168 \cdot 5909 \\ \hline 170234 \cdot 3864 \end{array} \right.$$

$$\text{For } \cdot 9 \left\{ \begin{array}{r} 542 \cdot 1477272 \\ 9 \\ \hline 487 \cdot 9329 \end{array} \right.$$

For ·03 $\left\{\begin{array}{r} 542{\cdot}1477272 \\ 3 \\ \hline 16{\cdot}2644 \end{array}\right.$

For ·007 $\left\{\begin{array}{r} 542{\cdot}1477272 \\ 7 \\ \hline 3{\cdot}7950 \end{array}\right.$

For ·0005 $\left\{\begin{array}{r} 542{\cdot}1477272 \\ 5 \\ \hline {\cdot}2711 \end{array}\right.$

Sum of the products.

$$\begin{array}{r} 170234{\cdot}3864 \\ 487{\cdot}9329 \\ 16{\cdot}2644 \\ 3{\cdot}7950 \\ {\cdot}2711 \\ \hline 170742{\cdot}6498 \end{array}$$

This process may be contracted thus:

$$\begin{array}{r} 542{\cdot}1477272 \\ 5739413 \\ \hline 1626443182 \\ 54214773 \\ 21685909 \\ 4879329 \\ 162644 \\ 37950 \\ 2711 \\ \hline 170742{\cdot}6498 \end{array}$$

Notice that the fractional part of the multiplier is reversed as well as the integral, which we might have anticipated. This method of multiplication is useful even with non-recurring decimals.

Multiply 43·7842195 by 2·102736 to 4 places.

$$\begin{array}{r} 43{\cdot}78422 \\ 6372012 \\ \hline 875684 \\ 43784 \\ 876 \\ 306 \\ 13 \\ 2 \\ \hline 92{\cdot}0665 \end{array}$$

It now only remains to shew how to arrange the terms of the multiplication.

Rule I. In contracted Multiplication of Decimals, place the unit of the multiplier under the last place of decimals to be retained,

then reverse the multiplier, and if possible* make the multiplier overlap the multiplicand by one figure to the left, and the multiplicand overlap the multiplier by one figure to the right.

II. Multiply each figure of the multiplier by the figure next to the right above it; do not write down this result, but merely carry the "nearest ten," and proceed to multiply as usual.

III. Write the products under one another, placing the first figures retained in a vertical line.

IV. Add the several products, and mark off, counting from the right of their sum, the number of decimal places arranged for.

Multiply 43·7842195 by 2·102736.

43784	2195
210	2736
87568	4390
4378	42195
87	5684390
30	64895365
1	313526585
	2627053170
92·066	6545745520

Suppose that in this case only three decimal places were required, then all the work to the right of the vertical line would be wasted. How can this useless work be avoided?

Examine each separate product. The 8 to the immediate left of the vertical in the first line is obtained by multiplying 4 in the multiplicand by 2 in the multiplier; write down the multiplicand, putting this 2 under the 4, thus:

```
43·7842195
       2
----------
87·568
```

The 8 of the second line is obtained by multiplying 1 by 8. Again, writing the 1 under the 8, we get,

```
43·7842195
      12
----------
87·568
 4·378
```

* This can always be done by putting or supposing ciphers.

The 7 of the third line is obtained by multiplying 2 by 3, and carrying 1 from the 2 × 7. Writing the 2 under the 7, we get,

```
43·7842195
 2 012
──────────
87·568
 4·378
    87
```

The 0 of the next line is obtained by multiplying 7 by 4, and carrying 2 from 7 × 3. Similarly, the 1 in the next line is obtained by carrying from 3 × 4. The multiplication by 6 yielded no result in the first three places of decimals, and is therefore omitted. Arranging now the whole sum, we have,

```
       43·7842195           43·7842
     6372 012             472 012
     ─────────            ───────
       87568                87568
A       4378         B       4378
          87                   87
          30                   30
           1                    2
     ────────             ───────
      92·064               92·065
```

In sum A, the third place is very inaccurate, because we have lost the carriage in the addition. This is partly compensated for by observing the law for curtailing decimals given in § 2, as in sum B, and even then the result is inaccurate. If the third place be of importance, arrange for four places.

5·790684 × ·023056 to 5 places.

```
   Either,           or,
  5·790684        ·023056
 65 0320Q*        70975
 ─────────        ───────
 11581            11528
  1737             1614
    29              207
     3                1
 ─────────        ───────
 ·1335Q           ·1335Q
```

Ans. ·1335.

* The Q indicates the absent units' figure.

$634\cdot25\dot{7}6\dot{9} \times 12\cdot0\dot{6}$ to 7 places, with *exactness;* we therefore shall take 8 places.

Either,	or,
634·2576976977	12·06666666667
7666 666666021	779 6796752436
634257697698	724000000000
126851539539	36200000000
3805546186	4826666666
380554619	241333333
38055462	60333333
3805546	8446666
380555	724000
38055	108600
3806	8446
381	724
38	108
4	8
7653·37621889	1
	7653·37621885

Ans. 7653·3762189.

Note first the great discrepancy between the two eighth places, which, however, does not affect the seventh. This discrepancy is due to the fact that in the left-hand sum the number of over-estimates considerably exceeds that of the under-estimates, whilst in the right-hand sum the reverse is the case.

Secondly, in the left-hand sum the fourth and succeeding lines are derived by successively curtailing the third. Care must be taken not to derive any of these lines from its immediate predecessor, as this may cause an accumulation of inaccuracy.

A computer should at all times test his results by some rough estimate, and this is especially important in decimals, where a displacement of the point produces so vital a difference. The most fertile source of error in contracted multiplication is an erroneous arrangement of the factors, but such an error may be easily detected.

In the last example of the preceding page, 5 and a fraction is to be multiplied by ·02... ; the product must therefore be something over ·1, as $5 \times \cdot02 = \cdot1$. In the next example, upwards of 600 is taken more than 12 times; therefore the product must be more than 7200.

EXERCISE XXI.

By vulgar fractions and by decimals, correct to 5 places, find :

(1) $32\frac{5}{7} \times 1\frac{4}{11}$. (4) $1\frac{1}{640} \times \frac{1}{1024}$.

(2) $7\frac{2}{3} \times \frac{5}{13}$. (5) $6528\frac{1}{25000} \times 3794\frac{1}{128}$.

(3) $1\frac{4}{15} \times \frac{7}{8}$. (6) $\frac{11}{19000} \times \frac{4}{365}$.

By decimals only, working each question in two ways, by making each factor multiplier and multiplicand in turn :

(7) $\cdot 03\dot{7}9\dot{4} \times 5\cdot 0\dot{0}8\dot{4}$ to 5 places.

(8) $\cdot 111\dot{6} \times 43\cdot 74\dot{2}85\dot{3}$ to 8 places.

(9) $\cdot 9321875 \times 4\cdot 2688$ to 4 places.

(10) $\cdot 38 \times \cdot 04125$ to 3 places.

(11) $1\cdot 48279 \times \cdot\dot{3}$ to 5 places.

(12) $468\cdot\dot{1}\dot{2} \times 299\cdot 875$ to the nearest integer.

§ 5. DIVISION.

CASE I. By a power of 10. (Cf. p. 107.) Shift the decimal point as many places to the left as there are ciphers in the divisor, prefixing ciphers if necessary, and curtail the decimal thus obtained.

$\frac{5}{13} \div 100$ to 5 places.

$$\frac{5}{13} \div 100 = \cdot 38461\ldots \div 100 = \cdot 0038\underset{5}{\cancel{4}}\cancel{6}1. \qquad \textit{Ans.}\ \cdot 00385.$$

CASE II. By a divisor with few significant figures.

$40\cdot 12\dot{8}6\dot{3} \div 70000$ to 5 places.

$$40\cdot 12\dot{8}6\dot{3} \div 70000 = \cdot 004012\dot{8}6\dot{3} \div 7.$$

$$\begin{array}{r} 7\underline{)\ \cdot 00401\overset{3}{\cancel{2}}\cancel{8}} \\ \cdot 000573 \end{array} \qquad \textit{Ans.}\ \cdot 00057.$$

$395\cdot 4\dot{0} \div 56000$ to 6 places.

$$395\cdot\dot{4}\dot{0} \div 56000 = \cdot 395\dot{4}\dot{0} \div 56.$$

$$\begin{array}{r} 8\underline{)\ \cdot 395404} \\ 7\underline{)\ \cdot 049426} \\ \cdot 007061 \end{array} \qquad \textit{Ans.}\ \cdot 007061.$$

$6 \div 51700$ to 7 places.

$$6 \div 51700 = \cdot 06 \div 517.$$

```
517)·0600(·00011605
       830       1
       3130
        2800
```

Ans. ·0001161.

$\cdot\dot{6} \div 8213000$ to 12 places.

$$\cdot\dot{6} \div 8213000 = \cdot 000\dot{6} \div 8213.$$

```
8213)·00066666(·0000000811721
           9626
           14136
            59236
             17456
              10306
```

Ans. ·000000081172.

$5\cdot\dot{1}4285\dot{7} \div \cdot 613$ to 8 places.

$$5\cdot\dot{1}4285\dot{7} \div \cdot 613 = 5142\cdot\dot{8}5714\dot{2} \div 613.$$

```
613)5142·857142(8·389652761
       2388
        5495
         5917
          4001
           3234
            1692
             4668
              3775
                97
```

Ans. 8·38965276.

EXERCISE XXII.

(1) $743\cdot 5\dot{8}\dot{7} \div 10$, 1000, 1000000, to 8 places.
(2) $8\cdot 3\dot{7} \div 40$ to 5 places.
(3) $74 \div 35000$ to 6 places.
(4) $\cdot\dot{5}\dot{3} \div 30$ to 8 places.
(5) $\cdot\dot{7}2\dot{5} \div 4\cdot 31$ to 5 places.
(6) $\cdot\dot{1}\dot{8} \div \cdot 0007$ to 4 places.
(7) $47\cdot 3\dot{4}\dot{5} \div \cdot 01$, ·001, ·00001, to 5 places.
(8) $\cdot 05 \div 630$ to 10 places.
(9) $\cdot 28473 \div \cdot 00761$ to 4 places.
(10) $\cdot\dot{5}8\dot{1} \div \cdot 0009$ to 5 places.
(11) $\cdot\dot{3}7\dot{0} \div \cdot 028$ to 6 places.

(12) $2{\cdot}\dot{5} \div 2{\cdot}5$ to 5 places.
(13) $\cdot001 \div 44$ to 6 places.
(14) $6{\cdot}587 \div 19$, 1900, to 7 places.
(15) $6{\cdot}\dot{5}8\dot{7} \div {\cdot}19$, 1·9, 19, to 7 places.
(16) $6{\cdot}5\dot{8}\dot{7} \div 4{\cdot}35$, 8100, to 9 places.
(17) $6{\cdot}58\dot{7} \div 4{\cdot}35$, 8100, 4350, ·81, to 10 places.
(18) ${\cdot}\dot{5}3846\dot{1} \div 1{\cdot}86$ to 3 places.
(19) ${\cdot}\dot{5}3846\dot{1} \div 36$, 360, ·0036, to 6 places.
(20) $\dot{5}3846\dot{1} \div 45$, 4·5, ·45, 45000, to 8 places.
(21) ${\cdot}\dot{0}\dot{7} \div 48007{\cdot}8$ to 5 places.

CASE III. By a divisor with many significant figures.

Lemma* 1. By adding a figure to the right of any series of digits, two operations are performed; the series is multiplied by 10, and is increased by the number of units expressed by that figure; e.g. $48579 = (4857 \times 10) + 9$. Conversely, by cutting off a figure from the right of any series of digits, two operations are performed; the series is diminished by the number of units expressed by that figure, and is divided by 10; e.g. $4857 = (48579 - 9) \div 10$.

Lemma 2. In a long series of digits, it matters comparatively little what the particular figure added or cut off happens to be, since the multiplication and division by 10 are respectively of *much* more moment than the addition or subtraction of a quantity less than 10; e.g. calling 480936 480930, is a much less error, as compared with the quantities under consideration, than calling 48 40 would be. 48 exceeds 40 by $\frac{1}{5}$ of 40; but 480936 exceeds 480930 by $\frac{1}{80155}$ of 480930. If from a series of digits we successively strike off the right-hand figures, the importance of the subtraction continually increases.

It follows directly from what was said in page 109, that the same effect is produced on the quotient by dividing the divisor as by multiplying the dividend. If then, instead of successively "bringing down," i.e. adding figures to the right of the dividend, we cut off figures from the right of the divisor, the quotient will remain the same so long as the divisor has a large number of

* A Lemma is a proposition which is only used as subservient to the proof of another proposition.—De Morgan's Algebra.

figures. On this truth, in fact, depends the method of guessing the figures of the quotient in integral division.

$11\cdot\dot{4}28\dot{5} \div 3\cdot1415927$.

```
        Full form.                              Contracted form.
31415927) 114285285 | (3·6378136              31415927) 114285285 (3·6378136
           20037504 | 2          6                       20037504
            1187948 | 08                                  1187948
             245470 | 275                                  245470
   A          25558 | 7862                     B            25559
                426 | 04468                                   427
                111 | 885415                                  113
                 17 | 6376342                                  19
                  1 | 92967078, &c.                            —
```

The quotient in A is obtained by bringing down or adding figures to the successive remainders; that in B by cutting off figures from the divisor, multiplying, however, each time the figure just cut off by the new figure of the quotient to ascertain the carriage, which is always to be the nearest ten.

In form B, though the quotient remains accurate till the sixth place, the remainders (which are the preparation for future figures in the quotient) shew inaccuracies much earlier.

The last figure of the quotient in the case here given is unusually accurate. It is advisable not to begin to cut off till the digits in the divisor are two more than the number of figures of the quotient still required.

Example: $\cdot004\dot{2}3\dot{9} \div \cdot3278$ to 7 places.

```
    3278) 42·39 (·0129324
          9 612
          3 0563
C          10619
             785
             129
             —                    Ans. ·0129324.
```

$\cdot004\dot{2}3\dot{9} \div \cdot\dot{3}27\dot{8}$ to 7 places.

```
    32783278) 4239239 (·0129311
              960911
              305246
D              10197
                 362
                  34
                   1              Ans. ·0129311.
```

In C, only two figures could be spared from the divisor; we therefore had to obtain all but the last two figures of the quotient in the usual manner. In D, the divisor was made to consist of nine figures, to enable us to cut off at once. In fact, we had one figure (3) more than we actually used.

Compare the following: $46 \div \cdot 007\dot{5}\dot{1}$ to 3 places. The number of figures to be retained in the divisor depends not on the number of decimal places, but on the number of significant figures required in the quotient. It is therefore necessary to ascertain the *position* of the first of these significant figures. Place the decimal point in the divisor after its first significant figure, and shift it in the dividend the same number of places in the same direction, thus multiplying or dividing both by the same number. Thus we obtain, $46000 \div 7 \cdot \dot{5}\dot{1}$. Beginning to divide, we find the first figure to be 6 thousands. Thus we shall require in the quotient four integral figures and the three decimal places; in all, six more figures. We further require the two initial figures of the divisor, which must consequently consist of $4 + 3 + 2 = 9$ figures; but then we begin to cut off at once.

```
7·51515151)46000·0000(6120·9677
              909 0909        8
              157 5757
                7 2727
                  5091
                   582
                    56             Ans. 6120·968.
```

Wording: for first remainder—6, carry 1; 30, 31 and 9′ is 40, &c.; second remainder—5, carry 1; 2 and 7′ is 9, &c.; third remainder—2; 10 and 7′, &c.; and so on.

$\cdot 11 \div 1937 \cdot 4\dot{3}\dot{7}$ to 8 places. First place the point in the divisor after the 9, and move it two places to the left in the dividend also: $\cdot 0011 \div 19 \cdot 374\dot{3}\dot{7}$.*

Wording: for first figure of quotient—19 in 0, ·0′; in 0, 0′; in 1, 0′; in 11, 0′; in 110, 5′;

∴ the quotient begins ·00005, and we therefore require four figures after the ciphers, and must retain 6 figures in the divisor.

* Where the first significant figure of the divisor is 1, followed by a large digit, as here, it is better to take the first two figures as a trial divisor.

$$19\cdot\cancel{3}\cancel{7}\cancel{4}\cancel{8})\cdot00110000(\cdot0000567\cancel{7}\cancel{6}$$

```
      13128      8
       1504
        148
         13
```

Ans. ·00005678.

Rule: In contracted division of decimals:

First, shift the decimal point in both divisor and dividend the same number of places and in the same direction, so as to bring it in the divisor into the most convenient place for ascertaining the denomination of the first significant figure of the quotient; we are then able to determine the number of significant figures required in the quotient.

Secondly, retain in the divisor, if long enough, two more places than this required number of significant figures. If not long enough, obtain the earlier figures of the quotient by "bringing down," and begin to "cut off" when the figures of the divisor are two more than those of the quotient yet to be found. If long enough, begin to "cut off" at once.

Thirdly, multiply each figure as it is "cut off" by the new figure of the quotient to carry the nearest ten.

EXERCISE XXIII.

(1) $862 \div 41\cdot8\dot{1}7\dot{4}$ to 4 places.

(2) $437 \div 215\cdot253$ to 3 places.

(3) $6 \div \cdot1573$ to 3 places.

(4) $\cdot\dot{7}2\dot{6} \div \cdot\dot{0}47\dot{3}$ to 4 places.

(5) $\cdot0041\dot{6} \div \cdot08\dot{3}$ to 5 places.

(6) $1 \div \cdot\dot{1}23\dot{4}$ to 5 places.

(7) $54 \div \cdot0003\dot{7}\dot{1}$ to the nearest unit.

(8) $\cdot7\dot{2}8\dot{3} \div 4\cdot562$ to 5 places.

(9) $\cdot\dot{4}6153\dot{8} \div \cdot\dot{5}3846\dot{1}$ to 6 places.

(10) $\cdot0053 \div 72654$ to 8 places.

(11) $\cdot\dot{3} \div \cdot\dot{1}4285\dot{7}$ to 6 places.

By vulgar fractions and by decimals, correct to 5 places, find:

(12) $\frac{1}{3} \div \frac{1}{11}$; $\frac{4}{9} \div \frac{3}{7}$; $\frac{4}{9} \div \frac{2}{3}$; $7 \div \frac{1}{7}$; $\frac{1}{11} \div \frac{1}{3}$; $\frac{3}{7} \div \frac{4}{9}$; $\frac{2}{3} \div \frac{4}{9}$; $\frac{1}{7} \div 7$; $\cdot042 \div \frac{11}{300}$; $\frac{11}{300} \div \cdot042$.

(13) If the length of the year be reckoned at $365\frac{1}{4}$ days, instead of its true length, 365·242264 days, in what time will the error amount to 11 days, also to $2\cdot\dot{3}$ days?

CHAPTER V.

PROGRESSIONS.

§ 1. Arithmetical. It is required to add the following series:

2, 5, 8, 11, 14, 17, 20, 23. *Ans.* 100.

A series such as this, where each succeeding "term" is formed from the preceding one by adding or subtracting the same quantity, is called an Arithmetical Progression, and the quantity invariably added or subtracted is called the Common Difference of the terms.

If no longer or more complicated series were ever proposed than that given above, there would be no need to generalize on the subject; but let it be proposed to add, 1, $3\frac{5}{8}$, $6\frac{1}{4}$, $8\frac{7}{8}$, &c., to 1000 terms. It would be laborious to write out the whole series in order to perform the addition. Examine the first series given; under it, write the same series in reverse order, and add the terms two and two; thus,

$$\begin{array}{r}
2,\ \ 5,\ \ 8,\ 11,\ 14,\ 17,\ 20,\ 23\\
\underline{23,\ 20,\ 17,\ 14,\ 11,\ \ 8,\ \ 5,\ \ 2}\\
25,\ 25,\ 25,\ 25,\ 25,\ 25,\ 25,\ 25 = 8\times 25 = 200.
\end{array}$$

Hence the sum of the *two* series is 200; that of the one series is 100. The number 25 is obtained by adding the first and the last terms, or the second and the last but one, and so on; this 25 is multiplied by the number of terms, 8; and the product is divided by 2. The question now arises: Will this method hold for all arithmetical progressions?

The nature of an A.P. is such that the second term exceeds the first by the common difference, and the last but one falls short of the last by the same quantity; hence the sum of the first and last must be equal to that of the second and last but one, which again equals the sum of the third and last but two, since the third exceeds the second by as much as the last but two falls short of the last but one, and so on. Hence the sum of the double series is the sum of the first and last multiplied by the number of terms. The sum of the single series, then, will be found by multiplying the sum of the first and last terms by *half* the number of terms.

This conclusion is pictured by the following "formula:"

$$s = (a + l) \times \frac{n}{2},$$

where s stands for the sum of the series,
a „ first term,
l „ last term,
n „ number of terms.

By this formula we can sum a series where l, the last term, is known. But in the series, 1, $3\frac{5}{8}$, $6\frac{1}{4}$, &c., given above, l has yet to be found.

Let d represent the common difference; the first term is a; the second, $a + d$; the third, $a + 2 \times d$, &c. In other words, the series might be written thus:

1st term,	2nd term,	3rd term,	4th term,	5th term, &c.
a,	$a + d$,	$a + 2 \times d$,	$a + 3 \times d$,	$a + 4 \times d$, &c.,

where the number of d's added to a in each term is one less than the number of the term; thus the 20th term will be $a + 19 \times d$, and the 1000th term of the above series is $1 + 999 \times 2\frac{5}{8} = 2623\frac{3}{8}$. The sum of the series is then,

$$(1 + 2623\tfrac{3}{8}) \times \tfrac{1000}{2} = 2624\tfrac{3}{8} \times 500 = 1312187\tfrac{1}{2}.$$

The formula for the last, or nth term is, $l = a + (n - 1) \times d$.

Exercise XXIV.

(1) Find the sum of 1, 2, 3, &c., to 1000 terms.

(2) Find the seventieth odd number.

(3) Find the sum of 100 terms of 1, 3, 5, &c.

(4) „ 60 „ $3\frac{1}{2}$, $4\frac{3}{4}$, 6, &c.

(5) „ 75 „ ·16, ·18, ·2, &c.

(6) Find the eightieth term of 2·5, 2·75, 3, &c.

(7) If I invest in a Building Society 10*s.* a month for the 1st year, £1 a month for the 2nd year, £1. 10*s.* a month for the 3rd year, and so on, what will be my payment in the 10th year, and how much shall I have invested altogether at the end of the 10th year?

(8) If a stone fall through 16·1 ft. in the 1st second of time, 48·3 ft. in the 2nd second, 80·5 ft. in the 3rd second, and so on, how deep will be the shaft of a mine where a stone takes 7 seconds to reach the bottom?

9) Prove that in a descending series $l = a - (n-1) \times d$, and, as before, $s = (a + l) \times \frac{n}{2}$.

§ 2. Geometrical Series ascending. A series where each new term is formed from the preceding term by multiplication instead of addition is called a Geometrical Progression, and the number by which we multiply each time is called the "common ratio." Thus 5, 50, 500, &c., is a geometrical progression whose common ratio is 10.

The multiples of a number are in arithmetical, the powers in geometrical progression; e.g.,

0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30......81, &c., are in A.P.
1, 3, 9, 27, ...81, &c., are in G.P.

It is required to sum the series, 4, 12, 36, 108, &c., to 6 terms. The common ratio in this series is 3; therefore the series may be written:

1st term,	2nd term,	3rd term,	4th term,	5th term,	6th term,
4	4×3	4×3^2	4×3^3	4×3^4	4×3^5

whence we see that the power of the common ratio in each term is one less than the number of the term. Thus the 20th term would be 4×3^{19}; or if l be the nth term, and r the common ratio, $l = a \times r^{n-1}$.

Let s be the sum of the six terms, then

$$s = 4 + 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + 4 \times 3^4 + 4 \times 3^5$$

If this series is multiplied by the common ratio 3, we obtain,

$$3 \times s = 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + 4 \times 3^4 + 4 \times 3^5 + 4 \times 3^6$$

Subtracting s from $3 \times s$, we get $2 \times s$, and subtracting from

$$\begin{array}{r} 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + 4 \times 3^4 + 4 \times 3^5 + 4 \times 3^6 \\ 4 + 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + 4 \times 3^4 + 4 \times 3^5 \qquad\quad \\ \hline \end{array}$$

we find that the intermediate terms cancel one another, and we have to subtract 4 from 4×3^6. Hence,

$$2 \times s = 4 \times 3^6 - 4, \text{ and } s = \frac{4 \times 3^6 - 4}{2} = 1456. \qquad \textit{Ans. } 1456.$$

This conclusion is pictured by the following formula: $s = \dfrac{a \times r^n - a}{r - 1}$

where s stands for the sum of the series,

a	„	first term,
r	„	common ratio,
n	„	number of terms.

Exercise XXV.

Sum the series:

(1) 1, 3, 9, &c., to 8 terms.

(2) ·001, ·01, ·1, &c., to 10 terms.

(3) $5, 5^2, 5^3$, &c., to 5 terms.

(4) 1, 1·05, 1·05 × 1·05, &c., to 6 terms, correct to 4 places.

§ 3. Geometrical Series descending. In a G. P. where the common ratio is an aliquot* fraction, say $\frac{1}{r}$, each term will be less than the preceding term, and the G.P. may be considered as one formed by division by r, instead of multiplication by $\frac{1}{r}$. Consider the series, $s = 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$.

Here each term is formed by dividing the preceding term by 3. If the series be multiplied by 3 (the common divisor), each term will become the one preceding it, and we obtain, calling s the sum of the above series,

$3 \times s = 27 + 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}$. Subtracting,
$1 \times s = \quad 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$, we obtain,
$(3 - 1) \times s = 2 \times s = 27 - \frac{1}{81} = 26\frac{80}{81}$, and $\therefore s = 13\frac{40}{81}$.

Exercise XXVI.

Sum the series:

(1) $1, \frac{1}{3}, \frac{1}{9}$, &c., to 8 terms.

(2) $1, \frac{1}{8}, \frac{1}{8^2}$, &c., to 5 terms.

(3) 1000, 100, 10, &c., to 10 terms.

(4) 3, ·3, ·03, &c., to 10 terms.

* An aliquot fraction is a fraction which is a measure of unity, and its reciprocal is consequently an integer.

§ 4. Endless Geometrical Series.

Examine the series, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$

$$\begin{array}{ll}
1 = 1 & = 2 - 1 \\
1 + \frac{1}{2} = 1\frac{1}{2} & = 2 - \frac{1}{2} \\
1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4} & = 2 - \frac{1}{4} \\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1\frac{7}{8} & = 2 - \frac{1}{8} \\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16} & = 2 - \frac{1}{16} \\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 1\frac{31}{32} & = 2 - \frac{1}{32} \text{ and so on.}
\end{array}$$

Let us apply the method of § 3 to the series, $1 + \frac{1}{2} + \frac{1}{4} + \&c.$ We must stop somewhere. Let us stop at $\frac{1}{32}$.

$$\begin{array}{rl}
s = & 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \\
2 \times s = & 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \\
\hline
s = & 2 - \frac{1}{32}, \text{ as above.}
\end{array}$$

Each successive series, then, falls short of 2 by its last term; but this last term continually diminishes, and can be made smaller than any assigned quantity. The series is therefore an approximation whose LIMIT is 2 (p. 119).

Again, take the series,

$$\begin{array}{rl}
s = & 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \mid + \&c. \\
3 \times s = & 27 + 9 + 3 + 1 + \dots\dots\dots \frac{1}{27} \mid + \&c. \\
\hline
(3 - 1) \times s = & 2 \times s = 27 - \frac{1}{81}
\end{array}$$

We see that $2 \times s$ falls short of 27 by the last term, and hence, since again the last term may be made less than any assigned quantity, the limit of $2 \times s$ is 27, and of s it is $13\frac{1}{2}$.

Find the limit of $7 + 1\frac{2}{5} + \frac{7}{25} + \&c.$ Here $\frac{1}{r} = \frac{1}{5}$ or $r = 5$.

$$\begin{array}{rl}
5 \times s = & 35 + 7 + 1\frac{2}{5} + \dots\dots \\
1 \times s = & \quad\;\;\, 7 + 1\frac{2}{5} + \dots\dots \\
\hline
\end{array}$$

$4 \times s = 35 -$ (some quantity which may be made as small as we please). Hence 35 is the limit of $4 \times s$, and the limit of $s = 8\frac{3}{4}$.

Exercise XXVII.

Find the limits of:

(1) $1 + \frac{1}{5} + \frac{1}{5^2} + \&c.$

(2) $3\frac{1}{2} + \frac{1}{2} + \frac{1}{14} + \&c.$

(3) $12 + 3 + \frac{3}{4} + \&c.$

(4) $64 + 8 + 1 + \&c.$

(5) $\cdot 5 + \cdot 25 + \cdot 125 + \&c.$

(6) $20 + 6\frac{2}{3} + 2\frac{2}{9} + \&c.$

§ 5. Limit of Recurring Decimals.

Every recurring decimal is a G.P. where r is a power of 10; thus, $\cdot\dot{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$, r being 10; $\cdot\dot{4}6\dot{8} = \frac{468}{1000} + \frac{468}{1000000} + \dots$, r being 1000.

Case I. Find the limit of $\cdot\dot{3}$.

$$\begin{aligned} s &= \cdot 333\ldots\ldots \\ 10 \times s &= 3\cdot 333\ldots\ldots \\ \hline 9 \times s &= 3 \qquad s = \tfrac{3}{9} = \tfrac{1}{3}. \end{aligned}$$

Ans. $\frac{1}{3}$.

Find the limit of $\cdot\dot{4}6\dot{8}$.

$$\begin{aligned} s &= \cdot 468468468\ldots\ldots \\ 1000 \times s &= 468\cdot 468468468\ldots\ldots \\ \hline 999 \times s &= 468 \qquad s = \tfrac{468}{999}\Big|\tfrac{52}{111}. \end{aligned}$$

Ans. $\frac{52}{111}$.

Exercise XXVIII.

Find the limits of the following:

(1) $\cdot\dot{3}$.	(7) $\cdot\dot{2}\dot{7}$.	(13) $\cdot\dot{1}4285\dot{7}$.	(19) $\cdot\dot{0}136986\dot{3}$.
(2) $\cdot\dot{7}$.	(8) $\cdot\dot{7}\dot{2}$.	(14) $\cdot\dot{8}5714\dot{2}$.	(20) $\cdot\dot{0}7936\dot{5}$.
(3) $\cdot\dot{1}$.	(9) $\cdot\dot{1}3\dot{5}$.	(15) $\cdot\dot{4}2857\dot{1}$.	(21) $\cdot\dot{0}028\dot{7}$.
(4) $\cdot\dot{2}$.	(10) $\cdot\dot{2}3\dot{4}$.	(16) $\cdot\dot{1}5384\dot{6}$.	(22) $\cdot\dot{0}1098\dot{9}$·
(5) $\cdot\dot{6}$.	(11) $\cdot\dot{0}2\dot{4}$.	(17) $\cdot\dot{1}44\dot{1}$.	(23) $\cdot\dot{0}421\dot{2}$.
(6) $\cdot\dot{9}$.	(12) $\cdot\dot{0}7\dot{4}$.	(18) $\cdot\dot{0}243\dot{9}$.	(24) $\cdot\dot{0}0025\dot{9}$.

It will be observed that the result uniformly must be this: the figures of the recurring period give the numerator, and the denominator consists of as many nines as there are recurring figures.

Case II. Find the limit of $\cdot 000\dot{2}\dot{7}$.

$$\begin{aligned} s &= \cdot 000272727\ldots\ldots \\ 100 \times s &= \cdot 027272727\ldots\ldots \\ \hline 99 \times s &= \cdot 027 \qquad s = \tfrac{\cdot 027}{99} = \tfrac{27}{99000}\Big|\tfrac{3}{11000}. \end{aligned}$$

Exercise XXIX.

Find the limits of:

(1) $\cdot 0\dot{3}$.	(3) $\cdot 00\dot{3}\dot{6}$.	(5) $\cdot 00\dot{1}221\dot{3}$.	(7) $\cdot 00\dot{1}0\dot{8}$.
(2) $\cdot 0\dot{7}\dot{2}$.	(4) $\cdot 0\dot{0}3\dot{6}$.	(6) $\cdot 0000\dot{9}$.	(8) $\cdot 000\dot{0}10\dot{8}$.

It will be observed that the numerator consists of the recurring figures, and the denominator of as many nines as there are recurring figures, followed by the number of non-recurring ciphers in the decimal.

CASE III. Find the limits of $\cdot 48\dot{3}2\dot{4}$.

$$\begin{array}{r} s = \cdot 48324324324\ldots\ldots \\ 1000 \times s = 483\cdot 24324324324\ldots\ldots \\ \hline 999 \times s = 482\cdot 76 \end{array}$$

$$s = \frac{482\cdot 76}{999} = \frac{48276}{99900} \Big| \frac{5364}{11100} \Big| \frac{447}{925}. \qquad \textit{Ans. } \frac{447}{925}.$$

EXERCISE XXX.

Find the limits of the following :

(1) $\cdot 1\dot{3}\dot{6}$.	(5) $\cdot 04\dot{7}\dot{2}$.	(9) $\cdot 225\dot{9}$.	(13) $\cdot 91\dot{6}$.
(2) $\cdot 6\dot{2}\dot{7}$.	(6) $\cdot 06\dot{5}6\dot{3}$.	(10) $\cdot 58\dot{3}$.	(14) $\cdot 08\dot{3}$.
(3) $\cdot 4\dot{7}\dot{2}$.	(7) $\cdot 22\dot{5}\dot{9}$.	(11) $\cdot 41\dot{6}$.	(15) $\cdot 8\dot{3}$.
(4) $\cdot 47\dot{2}$.	(8) $\cdot 2\dot{2}5\dot{9}$.	(12) $\cdot 001\dot{6}$.	(16) $\cdot 5\dot{8}\dot{3}$.

§ 6. The question we have been solving in § 5 might have been stated thus : What vulgar fraction would have produced the given recurring decimal? And the result might have been obtained experimentally, thus :

$$\begin{array}{lll} \frac{1}{9} = \cdot\dot{1} & \therefore\ \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, & \text{\&c.} = \cdot\dot{2}, \cdot\dot{3}, \cdot\dot{4}, \text{\&c.} \\ \frac{1}{99} = \cdot\dot{0}\dot{1} & \therefore\ \frac{2}{99}, \frac{13}{99}, \frac{75}{99}, & \text{\&c.} = \cdot\dot{0}\dot{2}, \cdot\dot{1}\dot{3}, \cdot\dot{7}\dot{5}, \text{\&c.} \\ \vdots & & \\ \frac{1}{99999} = \cdot\dot{0}000\dot{1} & \therefore\ \frac{2}{99999}, \frac{7628}{99999}, & \text{\&c.} = \cdot\dot{0}000\dot{2}, \cdot\dot{0}762\dot{8}, \text{\&c.} \end{array}$$

In fact, in dividing 1000 by 999 we obtain quotient 1 and remainder 1, which, being again multiplied by 1000, yields the same quotient with the same remainder, and so on. Again, in dividing 43000 by 999, we obtain quotient 43, and remainder 43; hence $\frac{43}{999} = \cdot\dot{0}4\dot{3}$. Similarly, $\frac{285}{999} = \cdot\dot{2}8\dot{5}$; $\frac{4723}{9999} = \cdot\dot{4}72\dot{3}$, and so on; which leads to the conclusion stated at the end of Case I.

For Case II., since $\cdot 00\dot{7}5\dot{1} = \cdot\dot{7}5\dot{1} \div 100$, and $\cdot\dot{7}5\dot{1} = \frac{751}{999}$, $\therefore\ \cdot 00\dot{7}5\dot{1} = \frac{751}{999} \div 100 = \frac{751}{99900}$.

For Case III., $\cdot 84\dot{3}5\dot{1} = \cdot 84 + \cdot 00\dot{3}5\dot{1} = \frac{84}{100} + \frac{351}{99900}$ $= \frac{999 \times 84}{99900} + \frac{351}{99900} = \frac{84 \times 1000 - 84 + 351}{99900} = \frac{84351 - 84}{99900} = \frac{84267}{99900}$.

This leads to the following general rule: For the numerator, subtract the non-recurring figures from the decimal, as given to the end of the first period. For the denominator, write as many nines as there are recurring figures, followed by as many ciphers as there are non-recurring figures.

CHAPTER VI.

PROPERTIES OF DECIMALS.

§ 1. We must premise that the vulgar fractions on which we shall reason in this Chapter are all at lowest terms, unless the contrary is stated.

§ 2. In Chaps. II. and III., all the decimal fractions were terminating; the subsequent Chapter treated of non-terminating decimals. The question arises: Can we, by mere inspection of a given vulgar fraction, determine the nature of the resulting decimal?

Decimalize $\frac{5}{8}, \frac{13}{25}, \frac{3}{7}, \frac{19}{28}$.

UNITS. TENTHS. HUNDREDTHS. THOUSANDTHS.

$$a.\quad \frac{5}{8} = \frac{5\times\overset{5}{\cancel{10}}}{\underset{4}{\cancel{8}}} = \frac{5\times5\times\overset{5}{\cancel{10}}}{\underset{2}{\cancel{4}}} = \frac{5\times5\times5\times\overset{5}{\cancel{10}}}{\underset{1}{\cancel{2}}} = \cdot 625.$$

$$b.\quad \frac{13}{25} = \frac{13\times\overset{2}{\cancel{10}}}{\underset{5}{\cancel{25}}} = \frac{13\times2\times\overset{2}{\cancel{10}}}{\underset{1}{\cancel{5}}} = \quad . \quad . \quad . \quad . \quad \cdot 52.$$

$$c.\quad \frac{3}{7} = \frac{3\times10}{7} = \frac{3\times10\times10}{7} = \frac{3\times10\times10\times10}{7} = \cdot 428\ldots$$

$$d.\quad \frac{19}{28} = \frac{19\times\overset{5}{\cancel{10}}}{\underset{14}{\cancel{28}}} = \frac{19\times5\times\overset{5}{\cancel{10}}}{\underset{7}{\cancel{14}}} = \frac{19\times5\times5\times10}{7} = \cdot 678\ldots$$

Fractions a and b give terminating decimals, because each successive introduction of 10 into the numerator gets rid of one 2 or one 5 in the denominator. If, then, the denominator has no other prime factors than 2 or 5, the decimal will terminate.

In fraction c, the denominator is prime to 10; hence no introduction of 10 into the numerator will cancel or reduce it.

In fraction d, the factors of the denominator are $2 \times 2 \times 7$, of which the 2×2 are cancelled by the two successive introductions of 10, but the 7 remains unaffected. From this we see that the solution of the question proposed depends solely upon the denominator.

Learn by heart: *If the denominator of a fraction contain no prime factors other than 2 or 5, its decimal will terminate; and if it contain others, it will not terminate.*

§ 3. Every introduction of 10 into the numerator will cancel 2, 5, or 2×5 in the denominator, and will yield one decimal place; hence there will be a decimal place for every 2×5 or 0 in the denominator, and a decimal place for every 2 or 5 after the ciphers are allowed for. (N.B. After accounting for the ciphers there cannot be twos *and* fives.)

§ 4. Notice that if the denominator, after disregarding any ciphers at the end, is a power of 2, the last figure of the decimal must be 5; if a power of 5, it must be even.

§ 5. We have now to consider the non-terminating fractions, and the question arises whether the figures will necessarily recur, or whether they will follow some other kind of arrangement, or no arrangement at all.

If in division by any number, say 7, 13, &c., any remainder occur a second time, the figures in the quotient, i.e. in the decimal, must thenceforward be the same as those following the previous occurrence of that remainder. Take, for example, $\frac{3}{7}$ and $\frac{39}{41}$.

7)30(·428571, &c.	41)390(·95121, &c.
20	210
60	50
40	90
50	80
10	39 &c.
3 &c.	*Ans.* $\cdot\dot{9}512\dot{1}$.
Ans. $\cdot\dot{4}2857\dot{1}$.	

In dividing by 7, there cannot be more than *six* different remainders, viz. 1, 2, 3, 4, 5, 6; in dividing by 41, there cannot be more than forty, viz. 1, 2, 3,......40. If, then, we should have performed respectively six or forty steps with different remainders, the next remainder must be one that has already occurred. Hence $\frac{3}{7}$ and $\frac{39}{41}$ must yield recurring decimals, and the recurrence must take place not later than after the sixth and fortieth places respectively. We see, however, from $\frac{39}{41}$, that it may take place earlier. Thus,

With denominator 7, recurrence will take place at latest after 6 places.
„ 41, „ „ 40 „
and generally, n, „ „ $(n-1)$ „

§ 6. If we take a number prime to 10, say 7, the L. C. M. of 10 and 7 is 70 (Part I. Ch. XI. § 18), and all tens below 70, viz. 10, 20, 30, 40, 50, 60, must give different remainders when divided by 7; for if any two gave the same remainder, their difference, a number of tens less than 70, would be a multiple of 7 (Part I. Ch. XI. 2nd part of § 7), which is impossible. These remainders, then, must be, in some order or other, the numbers 1, 2, 3, 4, 5, 6. Thus:

10	gives remainder	3,	40	gives remainder	5,
20	„	6,	50	„	1,
30	„	2,	60	„	4.

This means that each multiple of ten is a number of sevens + the corresponding number of units placed beside it. If, now, any two numbers of the first column, say 10 and 20, be multiplied together, we have to multiply (a number of sevens + 3) by (a number of sevens + 6); the multiplication by sevens gives sevens; the multiplication by 6 gives a number of sevens + 6×3. Similarly, the product of all these multiples of ten will be a number of sevens + the product of all the units to the right, or $10 \times 20 \times 30 \times 40 \times 50 \times 60$ = a number of sevens + $1 \times 2 \times 3 \times 4 \times 5 \times 6$. Hence, $10 \times 20 \times 30 \times 40 \times 50 \times 60 - 1 \times 2 \times 3 \times 4 \times 5 \times 6$ is a multiple of 7. But $10 \times 20 \times 30 \times 40 \times 50 \times 60 = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 1000000$, and $\therefore 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 1000000 - 1 \times 2 \times 3 \times 4 \times 5 \times 6$, or $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 999999$, must be a multiple of 7. Seven, then, must divide either $1 \times 2 \times 3 \times 4 \times 5 \times 6$ or 999999. (Part I. p. 147, note 2.) Suppose

the number we have chosen, as in this case 7, to be not only prime to 10, but a prime number, it cannot divide the first of these quantities, being prime to each of its factors; it must therefore divide the second, viz. 999999, which is formed by writing six nines in succession. Similar reasoning will shew that 13 will divide 999999999999 (twelve nines); 17 will divide sixteen nines; and, generally, if n is prime, n will divide $(n-1)$ nines in succession. (Fermat's Theorem.)

§ 7. Since 7 is a measure of 999999 (six nines), 13 of 999999999999 (twelve nines), and n* of $(n-1)$ nines, $\therefore$ 1000000 $(10^6) \div 7$ leaves remainder 1, $10^{12} \div 13$ leaves remainder 1, and $10^{n-1} \div n$* leaves remainder 1.

Again, 3000000 $(= 3 \times 10^6) \div 7$ leaves remainder 3,
$5 \times 10^{12} \div 13$ „ 5,
†$a \times 10^{n-1} \div n$* „ a.

Hence, in dividing by 7, 13, n, after 6, 12, $n-1$ places, we shall have as remainder the original dividend, and the quotient, i.e. the decimal fraction, will recur from the beginning, and have 6, 12, $n-1$ figures in the "period."

A prime number n, then (other than 2 or 5), will yield a pure circulator, having $n-1$ recurring figures. Thus $\frac{6}{7} = \cdot\dot{8}5714\dot{2}$ (6 figures), $\frac{5}{13} = \cdot\dot{3}8461538461\dot{5}$ (12 figures). This last decimal, however, can be indicated thus, $\cdot\dot{3}8461\dot{5}$ (6 figures); and generally, though n must yield a decimal *expressible* with $n-1$ recurring figures, it may be possible to use fewer figures in the period; but in this case this smaller number must consequently be a measure of $n-1$; or, in other words, though n must measure $n-1$ nines, it may measure an aliquot fraction of that number of nines.

§ 8. If $\frac{1}{7}$ be decimalized, we obtain $\cdot\dot{1}4285\dot{7}$, a period of six figures; hence all the remainders must have occurred. If, then, we wish to decimalize $\frac{2}{7}$, $\frac{3}{7}$, &c., we shall only have the same succession of figures with different commencements. Writing the period thus,

$$\begin{matrix} 7 & 1 & 4 \\ 5 & 8 & 2 \end{matrix}$$

we can from it read off the decimal for any number of sevenths.

* n being *prime*. † a being less than n.

Thus $\frac{2}{7}$ will begin with 2, $\frac{3}{7}$ with 4, $\frac{4}{7}$ with 5, &c., and this first figure can at once be ascertained by commencing the division. This ring should be learnt by heart.

Now decimalize $\frac{1}{13}$. $\frac{1}{13} = \cdot\dot{0}7692\dot{3}$.

$$\begin{array}{l} 13)\,100\,(\cdot\dot{0}7692\dot{3} \\ \quad\quad 90 \\ \quad\quad 120 \\ \quad\quad\quad 30 \\ \quad\quad\quad\; 40 \\ \quad\quad\quad\;\; 1, \text{ \&c.} \end{array}$$

This period contains six figures only, therefore only six of the 12 possible remainders can have occurred; and if we try another number of 13ths, we cannot be sure that that numerator will be one of these remainders. Take $\frac{2}{13}$.

$$\begin{array}{l} 13)\,20\,(\cdot\dot{1}5384\dot{6} \\ \quad\quad 70 \\ \quad\quad 50 \\ \quad\quad 110 \\ \quad\quad\; 60 \\ \quad\quad\;\; 80 \\ \quad\quad\;\;\; 2, \text{ \&c.} \end{array}$$

Here we find the other six remainders; and now no fraction whose denominator is 13 can be proposed whose numerator is not in one of the two sets of remainders. From what was said in § 5, it follows that the two series can at no point coincide. For 13, we shall then require these two rings:

$$\begin{array}{ccc} & 0 & \\ 3 & & 7 \\ 2 & & 6 \\ & 9 & \end{array} \qquad\qquad \begin{array}{ccc} & 1 & \\ 6 & & 5 \\ 4 & & 3 \\ & 8 & \end{array}$$

As these figures are not all different, in two of the cases the first *two* figures must be found by actual division. If there occur in any set of rings similar sequences of *two* figures each, the first *three* figures must be found by actual division, and so on.

§ 9. 365 days $= 5 \times 73$ days. Fractions with denominator 73 occur, therefore, frequently in the calculation of interest. 73 has eight recurring figures, and will therefore require $\frac{72}{8} = 9$ rings to exhaust the 72 remainders.

3	0	1		7	2	6		4	5	2		3	1	5		1	7	8
6	A	3		9	B	0		9	C	0		9	D	0		9	E	0
8	9	6		3	7	2		7	4	5		4	8	6		1	2	8

4	1	0		7	1	2		6	1	6		4	2	4
0	F	9		6	G	3		5	H	4		3	K	6
9	8	5		7	8	2		3	8	3		5	7	5

From these rings, any fraction with denominator 73 can be read off as a decimal, after the first two figures have been found by actual division.

§ 10. Examination of all these rings shews the peculiar property that the opposite numbers added together make 9. Thus the ring for 7 consists of the two halves 142 and 857, whose sum is 999. The question arises: Is this a mere accident or a property belonging to other than these fractions; and if so, to what others?

In Ch. V., we have found that every pure circulator can be expressed as a vulgar fraction having for denominator as many nines as there were recurring figures, and for numerator the recurring period. Thus $\frac{2}{7}$ or $\cdot\dot{2}8571\dot{4} = \frac{285714}{999999}$. This fraction is of course reducible to lower terms, where the new denominator is a measure of 999999. Now $999999 = 999 \times 1001$; seven being prime, it measures either 999 or 1001. We have already found that 7 gives six decimal places, i.e. that 999999 is the *least* number of nines divisible by 7; hence 7 does not measure 999, and must measure 1001. (Part I. p. 147, note 2). Since $\frac{285714}{999999} = \frac{285714}{999 \times 1001}$ will reduce to $\frac{2}{7}$, the factor 999 must be entirely cancelled out; hence the numerator must be divisible by 999. Let us, then, investigate the criterion for divisibility by 999.

$1000 \div 999$ leaves remainder 1
$7000 \div 999$ „ 7

$123325551 = 123000000 + 325000 + 551.$

$123000000 \div 999$	leaves remainder	123
$325000 \div 999$	„	325
$551 \div 999$	„	551
	Sum of the remainders,	999

Similarly 863584284267 ÷ 999 gives remainder 863 + 584 + 284 + 267 = 1998, and as this is divisible by 999, the number is so.

Hence a number is divisible by 999 (three nines) if the sum of its digits, added in sets of three figures beginning at the units' place, is divisible by 999.

Similar reasoning will shew that a number is divisible by 99, 9999, 99999, &c., if the sum of the digits, added up in sets of 2, 4, 5 figures respectively, be divisible by 99, 9999, 99999, &c. (Cf. the criterion for divisibility by 9 given in Part I. p. 132.)

Now we have shewn that 285714 must be divisible by 999, and therefore its digits added up in sets of three figures must be divisible by 999; and as we have but six figures, the sum of the two sets cannot exceed 999, and must therefore be 999. This can only take place if the opposite figures, as placed in the ring, make up together the number 9. (We shall call two figures whose sum is 9 complemental to 9.)

The conditions under which this process of reasoning holds good are, (*a*) that the number of recurring figures of the period is *even*, because then its corresponding nines can be broken up into two factors of which one is half the number of nines; (*b*) that the denominator of the vulgar fraction from which the recurring decimal is derived is prime to that less number of nines.

If, then, we know of any prime number that it will yield an *even* number of recurring figures, the halves of the period will be complemental, and accordingly we shall only need to work out by division the first half of the period. For example, of $\frac{19}{73}$, only four places, viz. 2602, need be found by division,

```
73)190(·2̇602|7397̇
     440
      200
       54
```

the other figures, 7397, being complemental. 11 has two recurring figures which must therefore be complemental, and only one needs to be found by division.

§ 11. The following table shews the number of recurring figures belonging to the first twenty-five primes, omitting 2 and 5.

PRIMES AND THEIR DECIMAL PLACES.

3...... 1	19... ..18	41...... 5	61......60	83......41
7...... 6	23......22	43......21	67......33	89... ..44
11...... 2	29......28	47......46	71......35	97......96
13...... 6	31......15	53......13	73...... 8	101...... 4
17......16	37...... 3	59......58	79......13	103......34

From this table we can extend the criteria for divisibility by prime numbers (Part I. Ch. XI. § 8); e.g., a number is divisible by 37 if the sum of its digits, added in sets of three beginning at the units' place, be divisible by 37; for 37 has three recurring figures, i.e. $1000 \div 37$ gives remainder 1, &c.

§ 12. Decimalize $\frac{63}{137}$.

$137)630(\cdot\dot{4}598540\dot{1}$
820
1350
1170
740
550
200
63

The successive dividends are,

63,	82,	135,	117
74,	55,	2,	20
137,	137,	137,	137

Notice that the eight dividends or remainders, corresponding to the eight recurring complementals, added in pairs as above, make up the divisor 137. This depends on the following considerations:

a. 99999999 is divisible by 137.

b. $99999999 = 9999 \times 10001$.

c. 137 does not measure 9999, for otherwise it would have only 4 recurring figures in its period; therefore, being prime, it must measure 10001.

d. $10001 \div 137$ leaves no remainder, $\therefore$ $10000 \div 137$ will leave a remainder 136, being short by 1 of the above multiple of 137.

e. $63 \times 10001 \div 137$ leaves no remainder, $\therefore$ $63 \times 10000 \div 137$ leaves remainder $137 - 63$, being 63 short of the multiple that 63×10001, or 630063, is of 137.

This means that the remainder after four steps must fall short of 137, the divisor, by 63, the first dividend. Had we begun with 82,

the second dividend, four steps would have led us to the same conclusion, ∴ after any four steps the dividend at the commencement added to the last remainder must give the divisor.

As this property depends on the same conditions as that investigated in § 10, it is subject to the same limitations. Hence, wherever the recurring figures are complemental to 9, the corresponding remainders will be complemental to the divisor.

§ 13. Reduce $\frac{11}{21}$ to a decimal.

$$\begin{array}{l} 21)110(\cdot\dot{5}2380\dot{9} \\ \quad\ \ 50 \\ \quad\ \ \ 80 \\ \quad\ \ \ 170 \\ \quad\ \ \ \ \ 200 \\ \quad\ \ \ \ \ \dot{1}10 \end{array}$$

Observe that this fraction, although 21 is not prime, yields a pure circulator. We might have judged, à priori, that such would be the case, for 21 is prime to 10, ∴ 210 is L.C.M. of 21 and 10, and no two multiples of 10 less than 210 can give the same remainder (§ 6); consequently any remainder, say 2, can only have been derived from the remainder 17, 17 only from 8, 8 only from 5, and 5 only from 11; hence the recurrence which must take place must be referable to the first dividend 11.

The statement in § 7, that a prime yields a pure circulator may now be extended to numbers prime to 10.

Again, observe that the recurring figures are not complemental to 9, nor the dividends to 21, 21 not being prime to 999.

§ 14. If the denominator contain as factors twos or fives with other primes, the resulting decimal will be a mixed circulator; for, dividing first by the twos or fives (§ 2), we shall obtain a terminating decimal; and if this is then divided by the remaining factor, which is prime to 10, we get a pure circulator commencing at the end of the terminating decimal previously obtained; e.g., $\frac{37}{52} = \frac{37}{4 \times 13} = \frac{9\cdot25}{13} = \cdot71\dot{1}5384\dot{6}$.

$$\begin{array}{l} \ \ \ 4)37 \\ 13)9\cdot25 \\ \quad\ \cdot71\dot{1}5384\dot{6} \end{array}$$

two non-recurring figures due to 4 (§ 3), followed by six recurring figures due to 13 (see table, § 11).

§ 15. It is now required to determine the number of recurring figures in the period derived from a vulgar fraction whose denominator, though composite, is prime to 10, or, which comes to the same thing, the least number of nines divisible by this denominator.

Decimalize $\frac{1}{7 \times 73} = \frac{1}{511}$. We know that 6 and 8 nines are the least number of nines divisible by 7 and 73 respectively. Hence 24 nines (24 being L. C. M. of 6 and 8) is the least number of nines divisible by 7 and 73, and $\therefore$ by 511,* or 511 will have 24 recurring figures.

Generally: A denominator which is a product of *different* primes other than 2 and 5 will have in the period as many recurring figures as is equal to the L. C. M. of the numbers of recurring figures proper to each prime. These conclusions do not apply to denominators containing as factors powers of prime numbers, the condition in the foot-note not being fulfilled.

[If p be a prime number with a period of k figures, then it can be proved that

p^2 will have a period of either		.... k or $p \times k$	figures,
p^3	„ „	...... k or $p \times k$ or $p^2 \times k$	„
p^4	„ „	k or $p \times k$ or $p^2 \times k$ or $p^3 \times k$	„
	&c.	&c.]	

§ 16. Addition and subtraction of recurring decimals, where the whole period of the result is required, need only be carried out to the number of places represented by the L. C. M. of the numbers of recurring figures in each of the given quantities + the maximum number of non-recurring decimals. An example will render this obvious.

Add $\cdot 58\dot{3}$, $\cdot 0\dot{0}41\dot{3}$, $\cdot 1234\dot{5}\dot{6}$, $\cdot\dot{1}5\dot{7}$.

$$\begin{array}{r|c|l}
\cdot 5833 & 333333333333 & 3 \\
\cdot 0041 & 304130413041 & 3 \\
\cdot 1234 & 565656565656 & 5 \\
\cdot 1571 & 571571571571 & 5 \\
\hline
\cdot 8680 & 774691883601 & \\
 & 1 \qquad\qquad\qquad 2 &
\end{array}$$

Ans. $\cdot 8680\dot{7}7469188360\dot{2}$.

* If a number is divisible by two numbers prime to each other, it is divisible by their product; and, conversely, if a number is divisible by a product, it must be divisible by each factor. (Part I. Ch. XI.)

The first vertical line divides the non-recurring from the recurring figures.

Under the first recurring figure the carriage is noted down, to be added to the last figure of the period, as the next column to the right of the second vertical line would have given the same carriage.

Subtract $\cdot 7\dot{3}5\dot{8}$ from $4\cdot 372\dot{1}$.

$$\begin{array}{r|r|l} 4\cdot 372 & 111 & 1 \\ \cdot 735 & 835 & 8 \\ \hline \multicolumn{3}{l}{3\cdot 636\ 27\cancel{6}} \\ \multicolumn{3}{l}{\qquad\ \ 1\ \ 5} \end{array}$$

Ans. $3\cdot 636\dot{2}7\dot{5}$.

In this case the carriage is subtracted.

§ 17. In Part I. (p. 3 and Ch. X.) we have seen that we are not necessarily confined to any one scale of notation, but that we might have chosen any number whatsoever for our radix. Similarly in fractions we need not necessarily make the powers of 10 our universal denominator. The powers of any other number might have been chosen, and the resulting fractions would have become continuations of the corresponding integral scales. Thus,

Decimal fractions are a continuation of the			decimal or denary scale
Binal	„	„	binary scale,
Quinal	„	„	quinary scale,
Duodecimal	„	„	duodecimal scale, &c.

Reduce $\frac{7}{16}$ to senals.

$\frac{7}{16} = \frac{1}{16}$ of $7 = \frac{1}{16}$ of $\frac{42}{6} = \frac{2}{6}$ and $\frac{10}{6}$ over.

$\frac{1}{16}$ of $\frac{10}{6} = \frac{1}{16}$ of $\frac{60}{36} = \frac{3}{36}$ and $\frac{12}{36}$ over.

$\frac{1}{16}$ of $\frac{12}{36} = \frac{1}{16}$ of $\frac{72}{216} = \frac{4}{216}$ and $\frac{8}{216}$ over.

$\frac{1}{16}$ of $\frac{8}{216} = \frac{1}{16}$ of $\frac{48}{1296} = \frac{3}{1296}$.

Ans. $\frac{2}{6} + \frac{3}{36} + \frac{4}{216} + \frac{3}{1296} = \dfrac{2}{6} + \dfrac{3}{6^2} + \dfrac{4}{6^3} + \dfrac{3}{6^4} = \cdot 2343.$*

Mod. op.: **7 and 16 in the senary scale = 11 and 24 respectively.**

$$\begin{array}{rlllllll} & 6\ \text{I.}\ \frac{1}{6} & & \text{I.}\ \frac{1}{6} & \frac{1}{6^2} & \frac{1}{6^3} & \frac{1}{6^4} \\ 24) & 1\ 1\cdot 0 & (& \cdot\ 2 & 3 & 4 & 3 \\ & 1\ 4\ 0 & & & & & \\ & \ \ 2\ 0\ 0 & & & & & \\ & \quad 1\ 2\ 0 & & & & & \end{array}$$

* Read, "point" 2343, and not "decimal" 2343.

Reduce $\frac{1}{7}$ to binals.

$$\begin{array}{r} \tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{8} \\ \text{Seven}\,)\ \underline{1\cdot 0\ 0\ 0} \\ \cdot 0\ 0\ 1\ldots \end{array}$$

Ans. $\cdot\dot{0}0\dot{1}$.

§ 18. We refrain from pursuing this subject further, but the curious are referred to Serré's Cours d'Arithmétique. We need only mention that the truths established for decimals hold, *mutatis mutandis*, for fractions in any scale, and that the several manipulations are identical. For example: the proof of Fermat's theorem given in § 6 holds for every scale of notation, substituting, however, for 999... repetition of the number which is 1 less than the radix.

§ 19. Summary.

Let $\frac{a}{b}$ be a fraction at lowest terms, and let it be reduced to a decimal.

(α) The character of the decimal will depend solely on b.

(β) If $b = 2^n$ or 5^n, or $2^n \times 5^m$, $\frac{a}{b}$ will terminate.

(γ) If $b = 2^n$ or 5^n, there will be n decimal places.

(δ) If $b = 2^n \times 5^m$, there will be n or m places according as n is greater or less than m.

(ϵ) If b is prime to 10, $\frac{a}{b}$ will give a pure circulator with a number of figures in the period, which number is invariable and depends solely on b.

(ζ) If b is a prime, this number is a measure of $b - 1$.

(η) If $b = 2^n \times c$, $5^n \times c$, or $2^n \times 5^m \times c$, c being prime to 10, the decimal will be a mixed circulator, having in the first and second cases n, and in the third n or m (whichever is greatest), non-recurring figures, followed by the number of recurring figures proper to c.

(θ) If b is prime to 10, and compounded of *different* primes, the number of recurring figures in the period is the L. C. M. of the numbers proper to these several primes.

(ι) If b is prime, and the number of figures in its period is even, the two halves of the period must be complemental to 9, and the corresponding remainders or dividends must be complemental to b.

CHAPTER VII.

DECIMALIZATION OF MONEY.

§ 1. We proceed to give a method of decimalizing our English money at sight, and to shew the numerous and great advantages accruing therefrom.

§ 2.

$$\begin{array}{rl}
2s.\ (1\text{ florin}) & =£{\cdot}1 \\
\therefore\ 4s. & =£{\cdot}2 \\
6s. & =£{\cdot}3 \\
\vdots & \quad\vdots \\
18s. & =£{\cdot}9
\end{array}$$

$$\begin{array}{rl}
1s. = & \tfrac{1}{2}\text{ of }2s. = \tfrac{1}{2}\text{ of }£{\cdot}1 \quad =£{\cdot}05 \\
\therefore\ 3s. = & 2s + 1s. = £{\cdot}1 + £{\cdot}05 = £{\cdot}15 \\
5s. = & 4s. + 1s. = £{\cdot}2 + £{\cdot}05 = £{\cdot}25 \\
7s. = & 6s. + 1s. = £{\cdot}3 + £{\cdot}05 = £{\cdot}35 \\
\vdots & \qquad\qquad\qquad\qquad\quad\vdots \\
19s. = & 18s. + 1s. = £{\cdot}9 + £{\cdot}05 = £{\cdot}95
\end{array}$$

$$\begin{array}{rlll}
6d. = & \tfrac{1}{2}\text{ of }1s. & = \tfrac{1}{2}\text{ of }£{\cdot}05 & = £{\cdot}025 \\
1s.\ 6d. = & 1s. + 6d. & = £{\cdot}05 + £{\cdot}025 & = £{\cdot}075 \\
2s.\ 6d. = & 2s. + 6d. & = £{\cdot}1 \ + £{\cdot}025 & = £{\cdot}125 \\
3s.\ 6d. = & 2s. + 1s.\ 6d. & = £{\cdot}1 \ + £{\cdot}075 & = £{\cdot}175 \\
\vdots & & & \quad\vdots \\
18s.\ 6d. = & 18s. + 6d. & = £{\cdot}9 \ + £{\cdot}025 & = £{\cdot}925 \\
19s.\ 6d. = & 18s. + 1s.\ 6d. & = £{\cdot}9 \ + £{\cdot}075 & = £{\cdot}975
\end{array}$$

Rule: The figure in the first decimal place will indicate the number of florins; for an odd shilling, add ·5 in the second place; for 6*d.* over, 25 in the second and third places; and for 1*s.* 6*d.* over, 75 in the second and third places.

Exercise XXXI.

Make a table of every sixpence from 6*d.* to 19*s.* 6*d.*

§ 3. The odd Farthings.

6*d.* being £·025, $\frac{1}{4}d. = {\cdot}025 \div 24.$

$$\begin{array}{r}
24\underline{)\ {\cdot}025\ } \\
{\cdot}001\tfrac{1}{24}
\end{array}$$

which means:

$$
\begin{array}{llll}
 & 1 \text{ farthing} & = £\cdot001 + \tfrac{1}{24} & \text{of } £\cdot001 \\
\therefore & 1 \quad ,, & = £\cdot001 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} & \text{of } £\cdot001^{*} \\
 & 2 \text{ farthings} & = £\cdot002 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} & \text{of } £\cdot002 \\
 & 3 \quad ,, & = £\cdot003 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} & \text{of } £\cdot003 \\
 & \vdots & \qquad\vdots & \\
5\tfrac{1}{2}d. = & 22 \quad ,, & = £\cdot022 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} & \text{of } £\cdot022 \\
5\tfrac{3}{4}d. = & 23 \quad ,, & = £\cdot023 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} & \text{of } £\cdot023
\end{array}
$$

Learn by heart: *Any number of farthings is the same number of thousandths of £1 + $\frac{1}{12}$ of $\frac{1}{2}$ of that number of thousandths.*

$$
\begin{aligned}
1\tfrac{1}{2}d. = 6 \text{ f.} &= £\cdot006 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot006 \\
&= £\cdot006 + \tfrac{1}{12} \text{ of } £\cdot003 \\
&= £\cdot006 + £\cdot00025 \\
&= £\cdot00625
\end{aligned}
$$

$$
\begin{aligned}
4\tfrac{1}{2}d. = 18 \text{ f.} &= £\cdot018 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot018 \\
&= £\cdot018 + \tfrac{1}{12} \text{ of } £\cdot009 \\
&= £\cdot018 + £\cdot00075 \\
&= £\cdot01875
\end{aligned}
$$

$$
\begin{aligned}
3\tfrac{1}{2}d. = 14 \text{ f.} &= £\cdot014 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot014 \\
&= £\cdot014 + \tfrac{1}{12} \text{ of } £\cdot007 \\
&= £\cdot014 + £\cdot00058\dot{3} \\
&= £\cdot014583
\end{aligned}
$$

$$
\begin{aligned}
1\tfrac{3}{4}d. = 7 \text{ f.} &= £\cdot007 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot007 \\
&= £\cdot007 + \tfrac{1}{12} \text{ of } £\cdot0035 \\
&= £\cdot007 + £\cdot000291\dot{6} \\
&= £\cdot0072916
\end{aligned}
$$

$$
\begin{aligned}
5\tfrac{1}{4}d. = 21 \text{ f.} &= £\cdot021 + \tfrac{1}{12} \text{ of } \tfrac{1}{2} \text{ of } £\cdot021 \\
&= £\cdot021 + \tfrac{1}{12} \text{ of } £\cdot0105 \\
&= £\cdot021 + £\cdot000875 \\
&= £\cdot021875
\end{aligned}
$$

Observe: The half of an even number of thousandths is obvious; for an odd number, put on 5 in the next place to the half of the even number below it. The student must learn to write down the result at once, passing *mentally* through the steps indicated above.

* This might have been derived from the pound directly, thus:

$$1 \text{ farthing} = £\tfrac{1}{960};\ £\tfrac{1}{960} > £\tfrac{1}{1000} \text{ by } £\tfrac{1}{960} - £\tfrac{1}{1000} = £\tfrac{1}{24000},$$

$$\therefore 1 \quad ,, \quad = £\cdot001 + \tfrac{1}{24} \text{ of } £\cdot001.$$

This very useful expression for a farthing also supplies the very curious one: 1 farthing $= £\cdot001 + \cdot01d.$

Wording: $1\frac{1}{2}d.=6$ f. $=£·0'0'6'$; $\frac{1}{2}$ of $6=3$; 12 in 30, $2'$, carry 6; in 60, $5'$.
Ans. £·00625.

$4\frac{1}{2}d.=18$ f. $=£·0'1'8'$; $\frac{1}{2}$ of $18=9$; 12 in 90, $7'$; in 60, $5'$. *Ans.* £·01875.

$3\frac{1}{2}d.=14$ f. $=£·0'1'4'$; $\frac{1}{2}$ of $14=7$; 12 in 70, $5'$; in 100, $8'$; in 40, $\dot{3}'$.
Ans. £·01458$\dot{3}$.

$1\frac{3}{4}d.=7$ f. $=£·0'0'7'$; $\frac{1}{2}$ of $7=35$; 12 in 35, $2'$; in 110, $9'$; in 20, $1'$; in 80, $\dot{6}'$. *Ans.* £·007291$\dot{6}$.

$5\frac{1}{4}d.=21$ f. $=£·0'2'1'$; $\frac{1}{2}$ of $21=105$; 12 in 105, $8'$; in 90, $7'$; in 60, $5'$.
Ans. £·021875.

$\frac{3}{4}d.=£·0'0'3'$; $\frac{1}{2}$ of $3=15$; 12 in 15, $1'$; in 30, $2'$; in 60, $5'$.
Ans. £·003125.

$\frac{1}{2}d.=£·0'0'2'$; $\frac{1}{2}$ of $2=1$; 12 in 10, $0'$; in 100, $8'$; in 40, $\dot{3}'$.
Ans. £·00208$\dot{3}$.

$\frac{1}{4}d.=£·0'0'1'$; $\frac{1}{2}$ of $1=5$; 12 in 5, $0'$; in 50, $4'$; in 20, $1'$; in 80, $\dot{6}'$.
Ans. £·001041$\dot{6}$.

$4d.=16$ f. $=£0'1'6'$; $\frac{1}{2}$ of 16, 8; 12 in 80, $\dot{6}'$. *Ans.* £01$\dot{6}$.

[$6d.=24$ f. $=£·0'2'4'$; $\frac{1}{2}$ of 24, 12; 12 in 12, $1'$. *Ans.* £·025.]

Exercise XXXII.

Make a table of every farthing from $\frac{1}{4}d.$ to $6d.$

§ 4. Decimalize $15s.\ 8\frac{1}{4}d.$

$$\begin{array}{rl} 15s.\ 6d. & =£·775 \\ 2\frac{1}{4}d. & =£·009 \\ \hline \end{array}$$
$$£·784+\tfrac{1}{12}\text{ of }\tfrac{1}{2}\text{ of }£·009=£·784375.$$

The three places ·784 can be obtained at once by adding the 9 thousandths from the farthings to the 75 thousandths from the shillings, and we have then only to add $\frac{1}{12}$ of $\frac{1}{2}d.$ of ·009.

Wording: $15s.\ 6d.$ is $·7'75$; $2\frac{1}{4}d.$ is 9 f., and 75, $8'4'$; $\frac{1}{2}$ of $9=45$; 12 in 45, $3'$; in 90, $7'$; in 60, $5'$. *Ans.* £·784375.

Decimalize $13s.\ 10\frac{3}{4}d.$

Wording: $13s.\ 6d.=£·6'75$; $4\frac{3}{4}d.=19$ f., $9'4'$; $\frac{1}{2}$ of 19, 95; 12 in 95, $7'$; in 110, $9'$; in 20, $1'$; in 8, $\dot{6}'$. *Ans.* £·694791$\dot{6}$.

Exercise XXXIII.

Decimalize:

(1) 8*s.* 6*d.*, 17*s.* 6*d.*, 13*s.* 6*d.*, 4*s.* 6*d.*, 1*s.* 6*d.*

(2) 13*s.* 3*d.*, 11*s.* 3*d.*, 19*s.* 3*d.*, 15*s.* 9*d.*, 18*s.* 9*d.*, 1*s.* 9*d.*

(3) 14*s.* $5\frac{1}{4}$*d.*, 11*s.* $10\frac{1}{2}$*d.*, 18*s.* $7\frac{1}{2}$*d.*, 3*s.* $11\frac{1}{4}$*d.*

(4) 8*d.*, 10*d.*, 7*d.*, $3\frac{1}{2}$*d.*, 1*s.* 1*d.*, 7*s.* 4*d.*

(5) 13*s.* 5*d.*, 17*s.* $10\frac{1}{4}$*d.*, 15*s.* $9\frac{1}{2}$*d.*, 13*s.* $8\frac{1}{2}$*d.*, 1*s.* $11\frac{1}{2}$*d.*, $\frac{1}{4}$*d.*, $\frac{1}{2}$*d.*, 1*d.*, 17*s.* $0\frac{1}{4}$*d.*, 11*s.* $0\frac{1}{2}$*d.*, 12*s.* $0\frac{1}{4}$*d.*

§ 5. Reconversion into Money.

1. The first decimal place gives florins.

2. A 5 (if any) in the second place gives 1*s.*

3. The remaining figures in the second and third places give each $\frac{24}{25}$ of a farthing; count them, then, as farthings, rejecting 1 if they exceed 24. The remaining figures of the decimal yield less than a farthing and may be disregarded.

Find the cost of 10, 100, 1000, 10000, 100000, and 1000000 articles, at 11*s.* $9\frac{3}{4}$*d.* each.

11*s.* $9\frac{3}{4}$*d.* = £·590625.

10 articles cost		£5·90625 = £5 + 9 fl. + 6 f. = £5. 18*s.* $1\frac{1}{2}$*d.*
100	,,	£59.0625 = £59 + 1*s.* + 12 f. = £59. 1*s.* 3*d.*
1000	,,	£590·625 = £590 + 6 fl. + 24 f. = £590. 12*s.* 6*d.*
10000	,,	£5906·25 = £5906 + 2 fl. + 1*s.* = £5906. 5*s.*
100000	,,	£59062·5 = £59062 + 5 fl. = £59062. 10*s.*
1000000	,,	£590625.

Find the cost of 10, 100, 1000, &c., articles at 3*s.* $10\frac{1}{4}$*d.* each.

3*s.* $10\frac{1}{4}$*d.* = £·1927083̇.

10 articles cost		£1·927083̇	= £1 + 9 fl. + 26 f. = £1. 18*s.* $6\frac{1}{2}$*d.*
100	,,	£19·27083̇	= £19 + 2 fl. + 1*s.* + 20 f. = £19. 5*s.* 5*d.*
1000	,,	£192·7083̇	= £192 + 7 fl. + 8 f. = £192. 14*s.* 2*d.*
10000	,,	£1927·083̇	= £1927 + 1*s.* + 32 f. = £1927. 1*s.* 8*d.*
100000	,,	£19270·833̇*	= £19270 + 8 fl. + 32 f. = £19270. 16*s.* 8*d.*
1000000	,,	£192708·333̇	= £192708 + 3 fl. + 32 f. = £192708. 6*s.* 8*d.*
10000000	,,	£1927083·333̇	= £1927083 + 3fl. + 32f. = £1927083. 6*s.* 8*d.*

and so on.

* The student is advised always to form three places, adding ciphers where necessary.

EXERCISE XXXIV.

Read off the cost of 10, 100, 1000, 10000, 100000, 1000000 articles at:

(1) 8s. 6d.
(2) 9s. $5\frac{1}{4}$d.
(3) £2. 17s. $4\frac{1}{2}$d.
(4) £3. 1s. $8\frac{1}{4}$d.
(5) 16s. $3\frac{3}{4}$d.
(6) £4. 13s. $3\frac{3}{4}$d.
(7) 2s. $7\frac{1}{4}$d.
(8) 3s. 7d.
(9) 4s. $8\frac{3}{4}$d.
(10) 5s. $10\frac{3}{4}$d.
(11) 6s. 4d.
(12) 1s. $5\frac{1}{2}$d.
(13) 4s. 2d.
(14) 6s. 1d.
(15) 8s. $0\frac{1}{4}$d.
(16) 12s. $0\frac{1}{2}$d.

The following values might with advantage be learned by heart:

£$\cdot\dot{3}$ = 6s. 8d.
£$\cdot\dot{6}$ = 13s. 4d.
£$\cdot 0\dot{3}$ = 8d.
£$\cdot 0\dot{6}$ = 1s. 4d.
£$\cdot 01\dot{6}$ = 4d.
£·01 = $\frac{1}{5}$ of £·05 = $\frac{1}{5}$ of 1s. = $2\frac{2}{5}$d.
£·02 = $\frac{2}{5}$ of 1s. = $4\frac{4}{5}$d.
£·03 = $7\frac{1}{5}$d.
£·04 = $9\frac{3}{5}$d.

§ 6. Find the cost of 5307 articles at £2. 9s. $4\frac{3}{4}$d. each.

£2. 9s. $4\frac{3}{4}$d. = £$2\cdot469791\dot{6}$.

```
  2·4697916    (3 places wanted.)
       7035
 ----------
   12348958
     740937
      17288
 ----------
  13107·183
```

Ans. £13107. 3s. 8d.

Find the cost of $2473\frac{13}{25}$ articles at £6. 5s. 8d. each.

$2473\frac{13}{25}$ = 2473·52.   £6. 5s. 8d. = £$6\cdot 28\dot{3}$.

```
   2473520          or      62833333
  33333826                    253742
  --------                  --------
  14841120                  12566667
    494704                   2513333
    197882                    439833
      7421                     18850
       742                      3142
        74                       126
         7                  --------
         1                 15541·951
  --------
 15541·951
```

Ans. £15541. 19s. $0\frac{1}{4}$d.

Find the dividend on £537. 8*s.* 10*d.* at 9*s.* $7\frac{3}{4}$*d.* in the £.

£$537{\cdot}441\dot{6} \times {\cdot}482291\dot{6}$ to 3 places.

```
   537·442
  2922 84
  ────────
   21 4977
    4 2995
      1075
       108
        48
         1
  ────────
   259·204
```

Ans. £259. 4*s.* 1*d.*

Exercise XXXV.

I.

(1) 3562 articles at 15*s.* $8\frac{3}{4}$*d.* each.
(2) 6019 „ £1. 2*s.* 10*d.* each.
(3) 7038 „ £3. 14*s.* $2\frac{3}{4}$*d.* each.
(4) 269 „ 3*s.* $10\frac{3}{4}$*d.* each.
(5) 5966 „ £7. 16*s.* 8*d.* each.
(6) 2469 „ £1. 10*s.* $10\frac{1}{4}$*d.* each.
(7) 9004 „ £2. 7*s.* $6\frac{1}{2}$*d.* each.
(8) 5040 „ 7*s.* $11\frac{1}{2}$*d.* each.
(9) 10010 „ 1*s.* $8\frac{3}{4}$*d.* each.
(10) 5039 „ 3*s.* $7\frac{1}{2}$*d.* each.

II.

(1) $843\frac{4}{5}$ articles at £3. 16*s.* $8\frac{3}{4}$*d.* each.
(2) $2047\frac{2}{3}$ „ £1. 17*s.* $2\frac{1}{2}$*d.* each.
(3) $3195\frac{5}{6}$ „ £2. 2*s.* $2\frac{1}{2}$*d.* each.
(4) 9843 doz. and 5 articles at 3*s.* $2\frac{1}{4}$*d.* per dozen.
(5) 7054 doz. and 11 „ £1. 9*s.* 10*d.* per dozen.
(6) 2437 doz. and 7 „ £2. 13*s.* $9\frac{1}{2}$*d.* per dozen.
(7) 5 years and 7 months at £31. 11*s.* 6*d.* a year.
(8) 10 years and 5 months at £9. 17*s.* 3*d.* a year.
(9) 17 years and 11 months at £10. 10*s.* 10*d.* a year.
(10) £1870. 16*s.* 8*d.* at 7*s.* $9\frac{3}{4}$*d.* in the £.
(11) £497. 10*s.* 4*d.* at 16*s.* $7\frac{1}{4}$*d.* in the £.

III.

(1) $2473\frac{13}{25}$ articles at £6. 5*s*. 8*d*. each.
(2) $649\frac{5}{7}$ " £1. 16*s*. 2*d*. each.
(3) $4037\frac{3}{11}$ " £2. 11*s*. $8\frac{1}{2}$*d*. each.
(4) $953\frac{13}{15}$ " £8. 7*s*. 10*d*. each.
(5) $2583\frac{7}{16}$ " £3. 3*s*. $11\frac{1}{2}$*d*. each.
(6) $7211\frac{7}{10}$ " £1. 18*s*. $7\frac{3}{4}$*d*. each.
(7) $2045\frac{4}{17}$ " £2. 2*s*. $9\frac{1}{2}$*d*. each.
(8) $477\frac{43}{73}$ " £4. 9*s*. 8*d*. each.
(9) 700700·07 " £10. 5*s*. 7*d*. each.
(10) $843594\frac{3}{4}$ " $6\frac{1}{2}$*d*. each.

§ 7. Decimalization of Other Fractions of a Penny.

Case I. Decimalize $9\frac{5}{8}d$.

$9\frac{5}{8}d. \times 2 = 1s.\ 7\frac{1}{4}d. = £\cdot080208\dot{3}$, $£\cdot080208\dot{3} \div 2 = £\cdot0401041\dot{6}$.

Ans. $£\cdot0401041\dot{6}$.

Case II. Decimalize $3\frac{11}{16}d$.

$3\frac{11}{16}d. \times 4 = 1s.\ 2\frac{3}{4}d. = £\cdot061458\dot{3}$, $£\cdot061458\dot{3} \div 4 = £\cdot015364958\dot{3}$.

Ans. $£\cdot015364958\dot{3}$.

Case III. Decimalize $2\frac{1}{32}d$.

$2\frac{1}{32}d. \times 8 = 1s.\ 4\frac{1}{4}d. = £\cdot067708\dot{3}$, $£\cdot067708\dot{3} \div 8 = £\cdot008463541\dot{6}$.

Ans. $£\cdot008463541\dot{6}$.

Case IV. Decimalize $7\frac{3}{10}d$.

$7\frac{3}{10}d. \times 10 = 6s.\ 1d. = £\cdot3041\dot{6}$, $£\cdot3041\dot{6} \div 10 = £\cdot03041\dot{6}$.

Ans. $£\cdot03041\dot{6}$.

Case V. Decimalize $5\frac{3}{5}d$.

$5\frac{3}{5}d. \times 5 = 2s.\ 4d. = £\cdot11\dot{6}$, $£\cdot11\dot{6} \div 5 = £\cdot02\dot{3}$. *Ans*. $£\cdot02\dot{3}$.

or,

$5\frac{3}{5}d. = 5\cdot6d.$, $5\cdot6d. \times 10 = 56d. = 4s.\ 8d. = £\cdot2\dot{3}$, $£\cdot2\dot{3} \div 10 = £\cdot02\dot{3}$.

Ans. $£\cdot02\dot{3}$.

Case VI. Decimalize $7\frac{13}{20}d$.

$7\frac{13}{20}d. \times 10 = 6s.\ 4\frac{1}{2}d. = £\cdot31875$; $£\cdot31875 \div 10 = £\cdot031875$.

Ans. £·031875.

Case VII. Decimalize $11\frac{4}{7}d$.

$11\frac{4}{7}d. \times 7 = 6s.\ 9d. = £\cdot3375$, $£\cdot3375 \div 7 = £\cdot0482\dot{1}4285\dot{7}$.

Ans. $£\cdot0482\dot{1}4285\dot{7}$.

Cases I., II. and III., are of common occurrence in commerce. Cases IV., V. and VI., are easy of calculation; for Case V. two methods are given, of which the second can be done mentally. Case VII. and the like are only to be found in examination papers.

Exercise XXXVI.

(1)	28493 lbs. of raw cotton at		$9\frac{11}{16}d.$
(2)	97058	,,	$1s.\ 0\frac{5}{8}d.$
(3)	247963	,,	$11\frac{19}{32}d.$
(4)	519766 lbs. of waste cotton at		$1\frac{25}{64}d.$
(5)	27964	,,	$3\frac{1}{5}d.$
(6)	24572	,,	$4\frac{4}{7}d.$
(7)	24572	,,	$2\frac{2}{5}d.$
(8)	967458	,,	$\frac{53}{64}d.$
(9)	1000000	,,	$1\frac{3}{7}d.$
(10)	257683	,,	$2\frac{7}{10}d.$
(11)	489573	,,	$4\frac{13}{20}d.$
(12)	359087	,,	$1s.\ 1\frac{11}{20}d.$

§ 8. Division of Money by Money.

In every case that actually occurs, the recurring figure in decimalized money will be 3 or 6,* which can be got rid of by multiplying by 3, for $3 \times \cdot\dot{3} = \cdot\dot{9} = 1$, $\therefore\ 3 \times \cdot\dot{6} = 2$.

Case I. £74. 8*s.* $4\frac{1}{2}d.$ ÷ £1. 9*s.* $6\frac{3}{4}d.$ to 5 places.

$$
\begin{array}{r}
1{\cdot}4\not{7}\not{8}\not{1}\not{2}\not{5})74{\cdot}41875(50{\cdot}34672 \\
{\cdot}512500 \\
69062 \\
9937 \\
1068 \\
33
\end{array}
$$

Ans. 50·34672.

or, stopping at the integral part of the quotient:

Ans. 50 times and £·5125 = 10*s.* 3*d.* over.

* 6*d.* = £·025.
$\frac{3}{4}d. = \frac{1}{8}$ of 6*d.*, terminating (Ch. VI. § 2).
$\frac{1}{4}d. = \frac{1}{3}$ of $\frac{3}{4}d.$, recurring, $\because$ ·025 is not divisible by 3.
$\frac{1}{2}d. = \frac{2}{3}$ of $\frac{3}{4}d.$,, ,,

If, then, the number of farthings over 6*d.* be divisible by 3, the decimal will terminate; if, when divided by 3, there be a remainder 1, the recurring figure is $\dot{3}$; if the remainder be 2, the recurring figure is $\dot{6}$.

CASE II. £58. 13*s.* 7*d.* ÷ 13*s.* $10\frac{1}{4}$*d.*

$$£58\cdot 6791\dot{6} \div £\cdot 692708\dot{3} = £58\cdot 6791\dot{6} \times 3 \div £\cdot 692708\dot{3} \times 3$$
$$= £176\cdot 0375 \div £2\cdot 078125.$$

$$\begin{array}{r} 2\cdot 078125)176\cdot 0375(84 \\ 9\cdot 7875 \\ 3)\underline{1\cdot 4750} \\ \cdot 491\dot{6} \end{array}$$

Ans. 84 times and 9*s.* 10*d.* over.

The remainder is divided by 3, in accordance with Ch. III. § 8, Case III.; or this may be done by contracted division at once, without multiplication by 3. (Ch. IV. p. 135, D.)

If in the divisor the fractional part of the penny be other than farthings, the denominator must be reduced by multiplication as in § 7.

EXERCISE XXXVII.

Work by decimals Exercise XX. in Part I.

CHAPTER VIII.

DECIMALIZATION OF WEIGHTS AND MEASURES.

§ 1. AVOIRDUPOIS WEIGHT.

a. Tons and cwts. at per ton.

1 ton : 1 cwt. = £1 : 1*s.* Hence call the cwts. shillings, decimalize and multiply.

Find the cost of 57 tons, 13 cwt., at £1. 5*s.* 9*d.* per ton.

57 tons, 13 cwt. (£57. 13*s.* = £57·65) = 57·65 tons. £1. 5*s.* 9*d.* = £1·2875.

$$\begin{array}{r} 12875 \\ \underline{5765} \\ 64375 \\ 90125 \\ 77250 \\ 64375 \\ \hline 74\cdot 224375 \end{array} \qquad \text{or} \qquad \begin{array}{r} 12\cdot 875 \\ \underline{5675} \\ 64375 \\ 9013 \\ 772 \\ 64 \\ \hline 74\cdot 224 \end{array}$$

Ans. £74. 4*s.* 6*d.*

b. Tons, cwts., qrs., at per ton.

1 ton : 1 qr. = £1 : 3*d.* Hence call every quarter 3*d.*, &c.

Find the cost of 17 tons, 11 cwt., 3 qrs., at £5. 2*s.* 6*d.* per ton.

17 tons, 11 cwt., 3 qrs. (£17. 11*s.* 9*d.*) = 17·5875 tons. £5. 2*s.* 6*d.* = £5·125.

```
17·5875
 5 215
-------
87 938
 1 759
   352
    88
-------
90·137
```

Ans. £90. 2*s.* 9*d.*

c. Tons, cwts., qrs., lbs., at per ton, the lbs. being a multiple of 7 lbs.

1 ton : 7 lbs. = £1 : $\frac{3}{4}$*d.* Hence call every 7 lbs. $\frac{3}{4}$*d.*

Find the cost of 19 tons, 5 cwt., 1 qr., 14 lbs., at 18*s.* 10$\frac{1}{2}$*d.* per ton.

19 tons, 5 cwt., 1 qr., 14 lbs. (£19. 5*s.* 4$\frac{1}{2}$*d.*) = 19·26875 tons.
18*s.* 10$\frac{1}{2}$*d.* = £·94375.

```
19·26875
573 49
--------
173 41
  7 70
    58
    13
     1
--------
18·183
```

Ans. £18. 3*s.* 8*d.*

Exercise XXXVIII.

Find the cost of:

(1) 43 tons, 17 cwt., at £5. 8*s.* 3*d.* per ton.

(2) 457 tons, 9 cwt., at £7. 10*s.* 10*d.* per ton.

(3) 8 tons, 1 cwt., at £1. 9*s.* 4$\frac{1}{2}$*d.* per ton.

(4) 9 tons, 8 cwt., 1 qr., at 16*s.* 8$\frac{3}{4}$*d.* per ton.

(5) 6 tons, 13 cwt., 2 qrs., at 17*s.* 9*d.* per ton.

(6) 7 tons, 5 cwt., 3 qrs., at 12*s.* 5*d.* per ton.

(7) 4 tons, 4 cwt., 1 qr., 7 lbs., at £43. 12*s.* 6*d.* per ton.

(8) 12 tons, 2 cwt., 2 qrs., 14 lbs., at £52. 10*s.* per ton.
(9) 23 tons, 3 cwt., 3 qrs., 21 lbs., at £24. 10*s.* 10*d.* per ton.
(10) 16 cwt., 1 qr., 14 lbs., at £18. 17*s.* 4*d.* per ton.

d. Tons, cwts., qrs., lbs., at per ton.

7 lbs. ($\frac{3}{4}d.$) = ·003125 tons.
1 lb. = $\frac{1}{7}$ of 7 lbs. = $\frac{1}{7}$ of ·003125 tons.
2 lbs. = $\frac{1}{7}$ of 2 × 7 lbs. = ($\frac{1}{7}$ of 2 × $\frac{3}{4}$ = $\frac{1}{7}$ of $1\frac{1}{2}d.$) = $\frac{1}{7}$ of ·00625 tons.
5 lbs. ($\frac{1}{7}$ of $3\frac{3}{4}d.$) = $\frac{1}{7}$ of ·015625 tons.
&c. &c.

Reduce 4 lbs. to the decimal of a ton.

4 lbs. ($\frac{1}{7}$ of 3*d.*) = $\frac{1}{7}$ of ·0125 = $\cdot 0017\dot{8}5714\dot{2}$.

Wording: 7 in 0, O′; in 1, O′; in 12, 1′; in $5\dot{5}$, 7′, and 6 over; $\frac{6}{7} = \cdot\dot{8}5714\dot{2}$. (See ring, Ch. VI. § 8.)

Express as tons, 13 cwt., 1 qr., 24 lbs.

13 cwt., 1 qr., 21 lbs. (13*s.* $5\frac{1}{4}d.$) = ·671875
3 lbs. ($\frac{1}{7}$ of $2\frac{1}{4}d.$) = $\frac{1}{7}$ of £·009375 = $\cdot 001339\dot{2}8571\dot{4}$

$\cdot 673214\dot{2}8571\dot{4}$

Ans. $\cdot 6732\dot{1}4285\dot{7}$ tons.

Find the cost of 5 tons, 7 cwt., 1 qr., 13 lbs., at £4. 7*s.* $8\frac{3}{4}d.$ per ton.

5 tons, 7 cwt., 1 qr., 7 lbs. (£5. 7*s.* $3\frac{3}{4}d.$) = 5·365625
6 lbs. ($\frac{1}{7}$ of $4\frac{1}{2}d.$) = $\frac{1}{7}$ of ·01875 = ·002678, &c.

5·368303 tons.

£4. 7*s.* $8\frac{3}{4}d.$ = $4\cdot 386458\dot{3}$.

5·368303

56 834

21 473
1 610
429
32
3

23·547 *Ans.* £23. 10*s.* $11\frac{1}{2}d.$

It will be observed that we have carried out the decimal further than was necessary. With a little practice the student will learn to avoid all such unnecessary labour.

Find the cost of 149 tons, 13 cwt., 3 qrs., 10 lbs., at £43. 8*s.* 4*d.* per ton.

149 tons, 13 cwt., 3 qrs., 7 lbs. (£149. 13*s.* $9\frac{3}{4}$*d.*) = 149·690625
3 lbs. ($\frac{1}{7}$ of $2\frac{1}{4}$*d.*) = $\frac{1}{7}$ of ·009375 = ·001339

149·691964 tons.

£43. 8*s.* 4*d.* = £43·41$\dot{6}$.

$$
\begin{array}{r}
149{\cdot}69196 \\
6666\ 1434 \\
\hline
598\ 7678 \\
44\ 9076 \\
5\ 9876 \\
1497 \\
898 \\
90 \\
9 \\
1 \\
\hline
6499{\cdot}125^{*}
\end{array}
$$

Ans. £6499. 2*s.* 6*d.*

Exercise XXXIX.

Find the cost of:

(1) 18 tons, 9 cwt., 1 qr., 16 lbs., at £3. 10*s.* 4*d.* per ton.
(2) 23 tons, 0 cwt., 2 qrs., 20 lbs., at £5. 4*s.* $7\frac{1}{2}$*d.* per ton.
(3) 93 tons, 7 cwt., 3 qrs., 8 lbs., at £6. 15*s.* 8*d.* per ton.
(4) 7 tons, 5 cwt., 1 qr., 21 lbs., at £10. 4*s.* per ton.
(5) 86 tons, 11 cwt., 2 qrs., 11 lbs., at £51. 17*s.* $4\frac{1}{2}$*d.* per ton.
(6) 17 cwt., 3 qrs., 26 lbs., at £9. 12*s.* 10*d.* per ton.
(7) 3 tons, 3 cwt., 3 qrs., 3 lbs., at £3. 3*s.* 3*d.* per ton.
(8) 2 tons, 14 cwt., 1 qr. 7 lbs., at £2. 14*s.* $3\frac{3}{4}$*d.* per ton.
(9) 5 cwt., 9 lbs., at £16. 16*s.* per ton.
(10) 4 tons, 1 qr., at £2. 5*s.* 8*d.* per ton.

e. cwts., qrs., and lbs., at per cwt.

1 cwt. : 1 qr. = £1 : 5*s.*, ∴ 1 qr. = ·25 cwt.
1 cwt. : 7 lbs. = £1 : 1*s.* 3*d.*, ∴ 7 lbs. = ·0625 cwt.
1 cwt. : 1 lb. = £1 : $\frac{1}{7}$ of 1*s.* 3*d.*, ∴ 1 lb. = ·0089$\dot{2}$8571$\dot{4}$.

* Comparison with the mode of working the same sum by Practice, Part II. p. 66, will justify the foot-note of p. 65. The difference in the number of figures used gives but a slight indication of the saving of ingenuity and labour, uniform simple multiplication being substituted for the intricate manipulation of fractions. The correctness of the work admits of an easy test by transposing the multiplier and multiplicand.

Find the cost of 3 tons, 13 cwt., 2 qrs., 19 lbs., at £1. 6*s.* 8*d.* per cwt.

73 cwt., 2 qrs., 14 lbs. (£73. 12*s.* 6*d.*) $= 73\cdot625$ cwt.

5 lbs. ($\frac{1}{7}$ of $5 \times 1s.\ 3d. = \frac{1}{7}$ of $6s.\ 3d.) = \frac{1}{7}$ of $\cdot3125 = \cdot044\dot{6}42857\dot{1}$

$73\cdot6696$, &c.

£1. 6*s.* 8*d.* $= £1\cdot\dot{3}$ or £$\frac{4}{3}$.

73·6696		
333·331		
73 670	or	73·6696
22 101		4
2 210		3)294·678
221		98·226
22		
2		
98·226		

Ans. £98. 4*s.* 6$\frac{1}{4}$*d.*

Exercise XL.

Find the cost of:

(1) 4 cwt., 1 qr., 17 lbs., at £3. 3*s.* per cwt.

(2) 3 qrs., 19 lbs., at £7. 8*s.* 4*d.* per cwt.

(3) 1 ton, 7 cwt., 3 qrs., 24 lbs., at £10. 12*s.* 8*d.* per cwt.

(4)*3 cwt., 2 qrs., 16 lbs., at £3. 7*s.* 8*d.* per cwt.

(5) 9 tons, 5 cwt., 16 lbs., at 12*s.* per cwt.

(6) 45 tons, 1 qr., 9 lbs., at £2. 7*s.* 10*d.* per cwt.

(7) 3 tons, 2 qrs., 8 lbs., at £5. 7*s.* 4*d.* per cwt.

(8) 3 cwt., 3 qrs., 12 lbs., at £1. 4*s.* 11*d.* per cwt.

(9) 15 tons, 15 cwt., 1 qr., 20 lb., at £10. 10*s.* per cwt.

f. lbs. and oz. at per lb.

1 lb. : 1 oz. = £1 : 1*s.* 3*d.*, ∴ 1 oz. = ·0625 lbs.

Ounces being binary fractions of the lb., are very easily decimalized, every two ounces being called 2*s.* 6*d.*

* N.B. 16 lbs. $= \frac{1}{7}$ of 1 cwt.; 2 qrs., 8 lbs. $= \frac{4}{7}$ of 1 cwt.
1 qr., 4 lbs. $= \frac{2}{7}$ of 1 cwt.; 2 qrs., 24 lbs. $= \frac{5}{7}$ of 1 cwt.
1 qr., 20 lbs. $= \frac{3}{7}$ of 1 cwt.; 3 qrs., 12 lbs. $= \frac{6}{7}$ of 1 cwt.
all of which can be decimalized at once from the ring, p. 148.

Express 11 oz. as a decimal of 1 lb.

11 oz. (11 × 1*s.* 3*d.* = 13*s.* 9*d.*) = ·6875 lbs.

Also 12 oz.

12 oz. (6 × 2*s.* 6*d.*) = ·75 lbs.

Exercise XLI.

Find the cost of:

(1) 3 qrs., 17 lbs., 5 oz., at 3*s.* 8*d.* per lb.

(2) 4 lbs., 14 oz., at £1. 17*s.* 6*d.* per lb.

(3) 9 oz., at £2. 15*s.* 4*d.* per lb.

(4) 15 lbs., 15 oz., at £6. 4*s.* 10½*d.* per lb.

(5) 1 cwt., 10 lbs., 6 oz., at 18*s.* 7½*d.* per lb.

(6) 9 lbs., 8 oz., at £2. 14*s.* 11*d.* per lb.

§ 2. Troy Weight.

a. lbs., oz., dwts. at per lb.
1 lb. : 1 oz. = £1 : 1*s.* 8*d.*
1 lb. : 1 dwt. = £1 : 1*d.* (Hence the name "*penny* weight.")
Hence call each oz. 1*s.* 8*d.*, and each dwt. 1*d.*

Find the cost of 9lbs., 3 oz., 14 dwt., at £10. 15*s.* 6*d.* per lb.

9 lbs., 3 oz., 14 dwt. (£9. 6*s.* 2*d.*) = $9{\cdot}308\dot{3}$ lbs. £10. 15*s.* 6*d.* = £10·775.

```
9·30833
5 7701
-------
9 3083
  6516
   651
    47
-------
100·297
```

Ans. £100. 5*s.* 11¼*d.*

As the lb. troy is hardly ever used, we proceed at once to the next case.

b. oz., dwts., grs., at per oz.
1 oz. : 1 dwt. = £1 : 1*s.*
1 oz. : 1 gr. = £1 : ½*d.*
Hence call each dwt. 1*s.*, and each gr. ½*d.*

Find the cost of a gold snuff-box weighing 7 oz., 15 dwts., 15 grs., at £4. 5*s.* 6*d.* per oz.

7 oz., 15 dwts., 15 grs. (£7. 15*s.* 7½*d.*) = 7·78125 oz. £4. 5*s.* 6*d.* = £4·275.

```
 7·78125
  5 724
 -------
31 125
 1 556
   545
    39
 -------
33·265
```

Ans. £33. 5*s.* 3¾*d.*

Exercise XLII.

(1) Find the cost of 17 lbs., 9 oz., 15 dwts., at £8. 4*s.* 11*d.* per lb.

(2) „ 9 lbs., 6 oz., 19 dwts., at £4. 19*s.* 4½*d.* per lb.

(3) „ 1 lb., 7 oz., 15 dwts., 20 grs. at £3. 17*s.* 10½*d.* per oz.

(4) „ 8 oz., 11 dwts., 17 grs., at £1. 14*s.* 10*d.* per oz.

(5) „ 10 oz., 10 dwts., 10½ grs., at £7. 11*s.* 4*d.* per oz.

(6) „ 4 lbs., 11 oz., 11 dwts., 11½ grs. at £10. 10*s.* per oz.

§ 3. Capacity.

a. gals., qts., pts. and gills, at per gal.

1 gal. : 1 qt. = £1 : 5*s.*
1 „ : 1 pt. = £1 : 2*s.* 6*d.*
1 „ : 1 gill = £1 : 7½*d.*

Hence call each quart 5*s.*, each pint 2*s.* 6*d.*, and each gill 7½*d.*

Find the cost of 5 gals., 3 qts., 1 pt., 1 gill, at £1. 5*s.* 6*d.* per gal.

5 gals., 3 qts., 1 pt., 1 gill (£5. 18*s.* 1½*d.*) = 5·90625 gals. £1. 5*s.* 6*d.* = £1·275.

```
5·90625
5 721
-------
5 906
1 181
  413
   30
-------
7·530
```

Ans. £7. 10*s.* 7¼*d.*

b. Quarters, bus., pecks, at per quarter.

1 qr. : 1 bus. = £1 : 2*s.* 6*d.*

1 qr. : 1 peck = £1 : 7½*d.*

Hence call each bushel 2*s.* 6*d.*, and each peck 7½*d.*

Find the cost of 43 qrs., 5 bus., 3 pecks, at £2. 10*s.* per qr.

43 qrs., 5 bus., 3 pks. (£43. 14*s.* 4½*d.*) = 43·71875 qrs.
£2. 10*s.* = £2·5

$$\begin{array}{r} 4\underline{)43{\cdot}71875} \\ 109{\cdot}296, \&c. = £109.\ 5s.\ 11\tfrac{1}{4}d. \end{array}$$

Exercise XLIII.

(1) Find the cost of 15 gals., 3 qts., 2 pts., 3 gills, at 10½*d.* per gal.

(2) „ 2 qts., 2 pts., 1 gill, at 1*s.* 4¾*d.* per gal.

(3) „ 29 gals., 1 qt., 3 pts., 2 gills, at 2*s.* 10½*d.* per gal.

(4) „ 428 qrs., 3 bus., 2 pecks, at £2. 13*s.* 6*d.* per qr.

(5) „ 617 qrs., 1 bus., 3 pecks, at £1. 3*s.* 9*d.* per qr.

§ 4. Length.

a. Yds., ft. and in., at per yard and per foot.

It is shortest to reduce the yards into feet and decimalize the inches by dividing by 12. If the price is given per ft., the decimal is at once available; if per yd., divide either the decimal or the price or the ultimate product by 3.

Find the cost of 8 yds., 1 ft., 11 in., at £1. 4*s.* 7*d.* per yd.

8 yds., 1 ft., 11 in. = $25{\cdot}91\dot{6}$ ft. = $8{\cdot}63\dot{8}$ yds. £1. 4*s.* 7*d.* = £$1{\cdot}2291\dot{6}$.

$$\begin{array}{r} 8{\cdot}6388 \\ \underline{29\ 221} \\ 8\ 639 \\ 1\ 727 \\ 173 \\ 77 \\ \underline{2} \\ 10{\cdot}618 \end{array}$$

Ans. £10. 12*s.* 4½*d.*

b. Miles, fur. and yds., at per mile.

$$1 \text{ m.} : 1 \text{ fur.} = £1 : 2s.\ 6d.$$
$$1 \text{ m.} : 11 \text{ yds.} = £1 : 1\tfrac{1}{2}d.$$

Hence call each furlong 2*s*. 6*d*., each 11 yds. $1\frac{1}{2}d.$, and each yd. $\frac{1}{11}$ of $1\frac{1}{2}d.$

Find the cost of 2 miles, 3 fur., 105 yds., at £15. 15*s*. per mile.

$$\begin{array}{lr}
2 \text{ m., } 3 \text{ fur., } 99 \text{ yds. } (£2.\ 8s.\ 7\tfrac{1}{2}d.) = & 2{\cdot}43125 \text{ m.} \\
6 \text{ yds. } (\tfrac{1}{11} \text{ of } 6 \times 1\tfrac{1}{2}d.) = \tfrac{1}{11} \text{ of } {\cdot}0375 = & \underline{{\cdot}00340\dot{9}\dot{0}} \\
 & 2{\cdot}43465\dot{9}\dot{0} \text{ m.}
\end{array}$$

$$£15.\ 15s. = £15{\cdot}75.$$

$$\begin{array}{r}
2{\cdot}434659 \\
\underline{5751} \\
2\ 4347 \\
1\ 2173 \\
1704 \\
\underline{122} \\
38{\cdot}346
\end{array}$$

Ans. £38. 6*s*. $11\frac{1}{4}d.$

c. Gunter's chain.

5 chains, $47\frac{1}{2}$ links, at 7*s*. $8\frac{1}{2}d.$ per ch.

$$\begin{array}{r}
5{\cdot}475 \times {\cdot}38541\dot{6} \\
3854 \\
\underline{5745} \\
1927 \\
154 \\
27 \\
\underline{2} \\
2{\cdot}110
\end{array}$$

Ans. £2. 2*s*. $2\frac{1}{4}d.$

Exercise XLIV.

(1) Find the value of 7 chains, $3\frac{3}{4}$ links, at 8*s*. $10\frac{1}{2}d.$ per chain.
(2) " 49 chains, $18\frac{1}{3}$ links, at 16*s*. $5\frac{1}{4}d.$ per chain.
(3) " 356 chains, $59\frac{1}{4}$ links, at £2. 14*s*. 8*d*. per chain.
(4) " 315 miles, 6 fur., at £768. 15*s*. per mile.
(5) " 19 yds., 2 ft., 8 in., at 7*s*. $8\frac{1}{3}d.$ per yd.
(6) " 43 yds., 1 ft., 11 in., at 2*s*. $5\frac{3}{4}d.$ per yd.
(7) " 1 mile, 4 fur., 77 yds., at £100 per mile.
(8) " 5 miles, 5 fur., 50 yds., at £20. 8*s*. per mile.

§ 5. SURFACE.

Acres, roods, poles, at per acre.

$$1 \text{ a.} : 1 \text{ r.} = £1 : 5s.$$
$$1 \text{ a.} : 1 \text{ p.} = £1 : 1\tfrac{1}{2}d.$$

Hence call each rood $5s.$, and each pole $1\frac{1}{2}d.$

Find the cost of 18 acres, 2 roods, 19 poles, at £46. 15*s.* per acre.

18 acres, 2 roods, 19 poles (£18. 12*s.* $4\frac{1}{2}d.$) = 18·61875 acres.

£46. 15*s.* = £46·75.

```
18·61875
   5764
--------
74 4750
11 1712
 1 3033
    931
--------
870·426
```

Ans. £870. 8*s.* $6\frac{1}{4}d.$

EXERCISE XLV.

(1) Find the cost of a close of 37 acres, 3 roods, 25 poles, at £42. 10*s.* per acre.

(2) " field of 19 acres, 1 rood, 35 poles, at £57. 15*s.* per acre.

(3) " farm of 368 acres, 2 roods, 30 poles, at £41. 15*s.* per acre.

(4) I bought 1572 acres, $1\frac{1}{2}$ roods, at £37. 17*s.* 6*d.* per acre, and of it sold of the best land 419 acres, $3\frac{1}{2}$ roods, at £47. 5*s.* an acre; how much an acre did the remainder stand me in?

§ 6. This method of translation into money would also be applicable to other weights and measures. Thus in paper measure, calling 1 ream £1, the quire is 1*s.*, and the sheet $\frac{1}{2}d.$; but we leave all further applications to the ingenuity of the student.

CHAPTER IX.

DECIMALS APPLIED TO PROPORTION.

Find the value of 37 yards of silk, when 25 yards cost £4. 7*s.* 6*d.*

$$\cancel{25} : 37 = 4{\cdot}375 : x$$
$$^{*}(\times 4)\ 100 \qquad 4$$

$$\begin{array}{r} 4{\cdot}375 \\ 4 \\ \hline 17{\cdot}5\cancel{0}\cancel{0} \\ 37 \\ \hline 1225 \\ 525 \\ \hline 6{\cdot}475 \end{array}$$

Ans. £6. 9*s.* 6*d.*

If a workman earns £17. 6*s.* in $102\frac{1}{2}$ days, how long will he be in earning 50 guineas?

$$17{\cdot}3 : 52{\cdot}5 = 102{\cdot}5 : x$$

$$\begin{array}{r} 102{\cdot}5 \\ 52\ 5 \\ \hline 512\ 5 \\ 2050 \\ 5125 \\ \hline 173)\,53812{\cdot}5\,(311 \\ 191\phantom{0{\cdot}5(311} \\ 182\phantom{{\cdot}5(311} \\ 9{\cdot}5 \end{array}$$

$$\frac{9{\cdot}5}{173} = \frac{95}{1730}\Big|\frac{19}{346}.$$ *Ans.* $311\frac{19}{346}$ days.

If the tax on £195 be £14. 8*s.*, what will be the tax on £874?

$$\cancel{195} : £874 = \cancel{14{\cdot}4} : x$$
$$65 \qquad 4{\cdot}8$$

$$\begin{array}{r} 874 \\ 6 \\ \hline 5244 \\ 8 \\ \hline 6\cancel{5})\,4195{\cdot}2\,(64{\cdot}541 \\ 295\phantom{{\cdot}2(64{\cdot}541} \\ 352\phantom{(64{\cdot}541} \\ 270\phantom{64{\cdot}541} \\ 10\phantom{64{\cdot}541} \end{array}$$

Ans. £64. 10*s.* 10*d.*

* Since $\frac{1}{4} = {\cdot}25$ and $\frac{3}{4} = {\cdot}75$, decimal fractions ending in 25 or 75 are often conveniently reduced to lower terms by multiplication by 4; similarly, decimals terminating in 125, 375, 625 and 875, are reducible by multiplication by 8. (Cf. Part. I. Ch. IX. § 19.)

If I can travel 198 miles by railway for £2. 9*s.*, how far at the same rate of charge ought I to be carried for £8. 0*s.* $10\frac{1}{2}d$?

$$
\begin{array}{rrr}
 & 2\cdot475 & : \; 8\cdot04375 = \cancel{198} : x \\
(\times 8) & \cancel{19}\cdot\cancel{8} & 8 \\
 & \cdot1 & 64\cdot35\cancel{000} \\
 & & 643\cdot5
\end{array}
$$

Ans. 643·5 miles.

The annual poor's-rates on a nett rental of £365. 7*s.* 3*d.* amount to £36. 8*s.* 9*d.* What should be the nett rental of an estate for which the poor's-rates amount to £24. 5*s.* 10*d.*?

$$
\begin{array}{rcrcrcl}
36\cdot4375 & : & 24\cdot291\dot{6} & = & 365\cdot3625 & : & x \\
8 & & 3 & & 8 & & \\
291\cdot5\cancel{000} & & 72\cdot875 & & 2922\cdot9\cancel{000} & & \\
3 & & 8 & & 58\;3 & & \\
874\cdot5 & & 583\cdot\cancel{000} & & 8768\;7 & & \\
8 & & & & 233832 & & \\
6996 & & & & 146145 & & \\
 & & & & 1704050\cdot7 & &
\end{array}
$$

$$
\begin{array}{l}
6996)\,1704050\cdot7\,(243\cdot575 \\
\quad 30485 \\
\quad\; 25010 \\
\quad\;\; 40227 \\
\quad\;\;\; 52470 \\
\quad\;\;\;\; 34980 \\
\quad\;\;\;\; \cdots\cdots
\end{array}
$$

Ans. £243. 11*s.* 6*d.*

Exercise XLVI.

Work, Part II., Ex. LI., Nos. 4, 7, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 53, 54, 63.

CHAPTER X.

PERCENTAGES.

§ 1. In commerce, the universal standard of comparison for estimating Profit and Loss, Premiums, Interest, Commission, Brokerage,

&c., is 100, whence the term *per centum*, shortened into *per cent.*, for which the symbol is %. Thus:

A profit of £5 on an *outlay* of £100 is called a profit of 5 %; similarly, a profit of 5*s.* on an outlay of 100*s.* is a profit of 5 %.

A premium of £2. 3*s.* 4*d.* to insure against the loss of £100 is called a premium of £2. 3*s.* 4*d.* %, or $2\frac{1}{6}$ %, or $2{\cdot}1\dot{6}$ %.

A brokerage of $\frac{1}{8}$ % means that the broker is to receive 2*s.* 6*d.* on every £100 transferred through him, &c.

Find the commission on £360 at 2 %.

$$\begin{array}{rcl} \text{If on £}100 & & \text{we pay £}2, \\ \text{on £}360 & & ,, \quad \text{£}x. \end{array}$$

$$\begin{array}{r} 100 : 360 = 2 : x \\ 3{\cdot}6 \times 2 = \quad 7{\cdot}2. \end{array}$$

Ans. £7. 4*s.*

Find the payment on £p at r %.

$$\begin{array}{rcl} \text{If on £}100 & & \text{we pay £}r, \\ \text{on £}p & & ,, \quad \text{£}x. \end{array}$$

$$100 : p = r : x$$

$$x = \frac{p \times r}{100}$$

or, if we call the payment £P, we obtain the "formula,"

$$^{*} P = \frac{p \times r}{100} \qquad \text{(I.)}$$

What is the sum to be paid for insuring a vessel and cargo worth £2225, at $3\frac{1}{4}$ %?

Here $p = £2225$, $r = 3{\cdot}25$, $\therefore \frac{p \times r}{100} = £22{\cdot}25 \times 3{\cdot}25$.

$$\begin{array}{r} 22{\cdot}25 \\ 3{\cdot}25 \\ \hline 11125 \\ 4450 \\ 6675 \\ \hline 72{\cdot}3125 \end{array} \qquad \text{or} \qquad \begin{array}{r} 22{\cdot}25 \\ 3\frac{1}{4} \\ \hline 66{\cdot}75 \\ 5{\cdot}5625 \\ \hline 72{\cdot}3125 \end{array}$$

Ans. £72. 6*s.* 3*d.*

* Note that P is the *payment* to be made, and p the *principal*.

What is the premium on £415 at £2. 3*s*. 4*d*. %?

	2·1667	or		$4·15\times2\frac{1}{6}$
	514			4·15
A	8 667		B	$2\frac{1}{6}$
	217			8·30
	108			·6916
	8·992			8·9916

Ans. £8. 19*s*. 10*d*.

The method A is universally applicable, but shorter methods, like B, will in certain cases suggest themselves to the computer.

What is the premium on £745 at £2. 8*s*. 7*d*. %?

$$\begin{array}{r} 2·4291\dot{6} \\ 547 \\ \hline 17\ 004 \\ 972 \\ 121 \\ \hline 18·097 \end{array}$$

Ans. £18. 1*s*. $11\frac{1}{2}d$.

Find the total gain, if goods bought for £357. 12*s*. 6*d*. are sold at a profit of 9 %.

$$\begin{array}{r} 3·57625 \\ 9 \\ \hline 32·186 \end{array}$$

Ans. £32. 3*s*. $8\frac{3}{4}d$.

Exercise XLVII.

(1) Find the insurance on £560 at $2\frac{3}{4}$ %.
(2) Find the insurance on £712. 10*s*. 8*d*. at $3\frac{5}{8}$ %.
(3) Find $5\frac{3}{8}$ % on £77. 7*s*. 7*d*.
(4) Find the brokerage on 750 guineas at $1\frac{1}{4}$ %.
(5) Find the brokerage on 4593 doz. at 5*s*. $10\frac{1}{2}d$. per doz. at $\frac{7}{8}$ %.
(6) Find the commission on £2595 at $2\frac{1}{2}$ %.
(7) Find the premium of fire insurance on £1550 at 3*s*. 4*d*. %.
(8) Find the profit realized on £1470 at $12\frac{1}{2}$ %.
(9) Find the loss on goods bought for £370 and sold at a loss of $7\frac{3}{8}$ %.
(10) Goods bought for £275. 13*s*. 10*d*. are sold at a profit of $15\frac{7}{8}$ %. What were they sold for?

(11) 17,500 dozen handkerchiefs were bought at 4*s.* $4\frac{1}{2}d.$ per dozen, and sold at a profit of $17\frac{3}{4}$ %. How much were they sold for?

(12) A population of 357,600 increased $3\frac{1}{2}$ % in a certain year; the deaths were 11,920. Find the number of births.

§ 2. Interest.

Interest is money paid for the loan of money, and is calculated at so much % per annum. The calculation is rendered more complex than the questions in § 1 by the introduction of the element of Time. The difference between the two is that between Simple and Compound Proportion.

Find the interest on £340 at 4 % (per annum) for 3 years.

If on £100 for 1 year we pay £4,
on £340 „ 3 years „ £x.

$$\left.\begin{matrix}100 : 340 \\ 1 : 3\end{matrix}\right\} = 4 : x.$$

$$3 \cdot 4 \times 12 = 40 \cdot 8. \qquad \textit{Ans.}\ £40.\ 16s.$$

Find the interest on £p at r % for t years.

If on £100 for 1 year we pay £r,
on £p „ t „ £x.

$$\left.\begin{matrix}100 : p \\ 1 : t\end{matrix}\right\} = r : x.$$

$$x = \frac{p \times r \times t}{100} = \text{Interest.} \qquad \textit{Ans.}\ I = \frac{p \times r \times t}{100}*. \quad \text{(II.)}$$

Find the interest on £463. 14*s.* 9*d.* at $2\frac{3}{4}$ % for 1 year, 5 months.

Here $p = £463 \cdot 7375$, $r = 2 \cdot 75$, $t = 1\frac{5}{12} = 1 \cdot 41\dot{6}$,

$$\therefore \frac{p \times r \times t}{100} = £4 \cdot 637375 \times 2 \cdot 75 \times 1 \cdot 41\dot{6}.$$

$$\begin{array}{r} 4 \cdot 637375 \\ 572 \\ \hline 9\ 2747 \\ 3\ 2461 \\ 2319 \\ \hline 12 \cdot 7527 \end{array} \qquad \text{(4 places.)}$$

* The money lent is called the *principal;* the yearly sum to be paid for £100, the *rate;* the interest + the principal = the *amount.* The letters p, r, t, I, have been chosen as the initials of the words principal, rate, time and interest.

12·7527 (3 places.)
766 141
12 753
5 101
128
76
7
1
18·066

Ans. £18. 1*s*. 4*d*.

or thus:

$2\frac{3}{4} \times 1\frac{5}{12} = \frac{11}{4} \times \frac{17}{12} = \frac{187}{48} = 3·8958\dot{3}.$

4·637375 (3 places.)
85 983
13 912
3 710
417
23
3
18·065 *Ans.* £18. 1*s*. 3¾*d*.

Find the interest on £212. 10*s*. 4*d*. for $2\frac{3}{4}$ years at $2\frac{1}{2}$ % per annum.

$2\frac{3}{4} \times 2\frac{1}{2} = \frac{55}{8} = 6·875$ or $2·1251\dot{6} \times 2·5 \times 2·75.$

$2·1251\dot{6}$
5 786
12 750
1 700
148
11
14·610

or

52
4 2503
1 0626
5·3129
572
1 0626
3718
266
14·610 *Ans.* £14. 12*s*. 2½*d*.

Experience alone will enable the computer to determine by inspection in which order it is best to take the factors.

Find the interest on £417. 7*s*. 9*d*. for 1 year, 10 months, at $4\frac{3}{8}$ %.

$4·173875 \times 1·8\dot{3} \times 4·375$ or $4·173875 \times \frac{35}{8} \times \frac{11}{6}.$

5734
166955
12521
2921
209
18·2606
333 381
18 261
14 608
548
55
5
33·477

or

$\frac{35}{8} \times \frac{11}{6} = \frac{385}{48} = \frac{96·25}{12} = 8·0208\dot{3}.$

4·1739
80 208
33391
83
3
33·477

Ans. £33. 9*s*. 6½*d*.

Find the interest on £713. 10*s.* 9*d.* at $4\frac{5}{8}$ % for 79 days.

$$7{\cdot}135375 \times 4{\cdot}625 \times \tfrac{79}{365}.$$

```
5)79                                  7·13537
73)15·8(·2i6438'3'5'6'*                  5264
    1 20                             ---------
     470                             28 5415
      320                             4 2812
       280                              1427
        61 &c.                           357
                                     ---------
                                     33·0011
                                     446 12
                                     ---------
                                     6 600
                                       330
                                       198
                                        13
                                         1
                                     ---------
                                     7·142
```

Ans. £7. 2*s.* $10\frac{1}{4}$*d.*

Exercise XLVIII.

I.

(1) Find the interest on £350 for 2 years at 5 %.
(2) „ „ £350 „ 5 „ 2 %.
(3) „ „ £765 „ $1\frac{1}{2}$ „ $2\frac{3}{4}$ %.
(4) „ „ £548. 16*s.* 3*d.* for 9 months at $4\frac{7}{8}$ %.
(5) „ „ £3456. 17*s.* 6*d.* for $4\frac{1}{2}$ „ $3\frac{3}{4}$ %.
(6) „ „ £279. 12*s.* 10*d.* for 6 „ $4\frac{3}{8}$ %.

II.

(1) Find the interest on £37. 18*s.* 9*d.* at $2\frac{7}{8}$ % for 47 days.
(2) „ „ £143. 10*s.* 8*d.* at $4\frac{5}{8}$ % for 100 days.
(3) „ „ £75. 16*s.* at $6\frac{1}{8}$ % for 42 days.
(4) „ „ 1000 guineas at 5 % for 12 days.
(5) „ „ £10490 at $4\frac{3}{4}$ % for 80 days.
(6) „ „ £876. 13*s.* 4*d.* at $3\frac{5}{8}$ % for 146 days.

* See Ch. VI. §§ 10 and 12; and note (*a*) that $61+12=73$; (*b*) that the accented figures are obtained at once by complementing 9. Or the period for $\frac{15}{73}$ might have been found at once from ring H, p. 150.

III.

(1) Find the interest on £1745. 12*s.* 8*d.* at £12. 15*s.* 8*d.* % for 130 days.

(2) „ „ £2495. 17*s.* 4*d.* at £13. 16*s.* 10*d.* % for 67 days.

(3) „ „ £65. 4*s.* 10*d.* at £5. 1*s.* 7¼*d.* % for 91 days.

(4) „ „ £147. 16*s.* 6*d.* at £7. 12*s.* 10½*d.* % for 39 days.

(5) „ „ £439. 10. 3*d.* at £9. 8*s.* 5*d.* % for 117 days.

(6) „ „ £455. 5*s.* 5*d.* at £8. 15*s.* 5¾*d.* % for 19 days.

Find the amount on: IV.

(1) £1483. 17*s.* 4*d.* for 1 year, 5 months, at 6¼ %.
(2) £1517. 16*s.* 2⅜*d.* for 4 years, 8 months, at 5 %.
(3) £2045. 3*s.* 10*d.* for 76 days at 3⅝ %.
(4) £439. 11*s.* 5*d.* for 91 days at 4⅔ %.
(5) £254. 8*s.* 7*d.* for 145 days at £5. 8*s.* 11¼*d.* %.
(6) £7777. 7*s.* 7*d.* for 77 days at £7. 7*s.* 7*d.* %.

V.

Find the interest for 1 minute on the national debt, £730,000,000, at 3 % per annum.

§ 3. Converse of Percentage.

We have seen (§ 1) that $P=\frac{p\times r}{100}$. Since $p \times r$ must be divided by 100 to obtain P, $100 \times P = p \times r$. Again, since p must be multiplied by r to yield $100 \times P$, p is the rth part of $100 \times P$, or

$$p=\frac{100\times P}{r} \qquad \text{(III.)}$$

Similarly,

$$r=\frac{100\times P}{p} \qquad \text{(IV.)}$$

If, then, of the three quantities, P, p and r, any two be given, the third can be found by means of the formulæ (I.), (III.), (IV.).

Find the brokerage on £475 at ¾ %.

$$P=\frac{p\times r}{100}=\frac{475\times\cdot75}{100}=4\cdot75\times\cdot75=3\cdot5625.$$

Ans. £3. 11*s.* 3*d.*

On what principal will the brokerage at $\frac{3}{4}$ % amount to £3. 11*s*. 3*d*.?

$$p=\frac{100\times P}{r}=\frac{100\times 3\cdot 5625}{\cdot 75}=\frac{356\cdot 25}{\cdot 75}=475. \qquad \textit{Ans.}\ £475.$$

At what rate will the brokerage on £475 amount to £3. 11*s*. 3*d*.?

$$r=\frac{100\times P}{p}=\frac{100\times 3\cdot 5625}{475}=\frac{356\cdot 25}{475}=\cdot 75. \qquad \textit{Ans.}\ \tfrac{3}{4}\ \%.$$

Exercise XLIX.

*(1) If goods are bought for £415 and sold for £500, what is the gain %?

(2) If the goods had been sold for £400, what would have been the loss %?

(3) On the breaking out of the war of 1870, the marine insurance of certain goods was raised from 2 to $5\frac{1}{2}$ %, making a difference of £140 in the premium to be paid. Find the value of the goods.

(4) A bought a horse for £40, and sold it to B at a profit of $8\frac{3}{4}$ %; B sold it at a loss of $7\frac{1}{2}$ % to C. What did C pay for the horse?

(5) What would C have paid if B had made $7\frac{1}{2}$ % profit?

(6) I bought French jewellery for fr. 7490, and paid an import duty of 15 % ad valorem; I sold it for £420. Find my gain or loss %, reckoning the £ at fr. 25·22.

(7) If $3\frac{3}{4}$ tons of sulphur are required to make 31 tons, 5 cwt. of gunpowder, what is the percentage of sulphur in gunpowder?

(8) At what rate will the brokerage on £1720. 16*s*. 8*d*. amount to £6. 9*s*. $0\frac{3}{4}$*d*.?

(9) On what sum will the brokerage at $\frac{5}{8}$ % amount to £10. 15*s*.?

(10) In a school of 80 children $17\frac{1}{2}$ % are girls. Find the number of boys.

(11) Find the rates to be paid on a house rented at 60 guineas and rated at 85 % of the rent, paying 1*s*. $3\frac{1}{2}$*d*. in the £.

(12) The total number of prisoners in a certain county was 493, of whom $\frac{17}{29}$ were males. The number of male prisoners in the following year was 272, and that of females $108\frac{1}{3}$ % of the former number. Find the total increase or decrease %.

* When not otherwise stated, find to 3 places.

(13) This year the number of out-door and in-door patients of a hospital were 1575 and 333 respectively; in the previous year the former were ·96 of this year's number, and the total was 1870. Find to one place the increase or decrease % of the in-door patients.

(14) A certain grocer's weekly business is as follows:

Goods.		Cost price.		Selling price.
5 cwt. of raw sugar	@	36*s*. per cwt.	@	4*d*. per lb.
$2\frac{1}{2}$,, loaf sugar	,,	44*s*. ,,	,,	$5\frac{1}{2}$*d*. ,,
$\frac{1}{2}$,, coffee	,,	84*s*. ,,	,,	1*s*. 6*d*. ,,
which loses 20 % in roasting, and costs further 3*d*. per lb. duty and 2*s*. 6*d*. per cwt. roasting.				
40 lbs. of tea	,,	2*s*. 8*d*. per lb.	,,	3*s*. 4*d*. ,,
$\frac{1}{2}$ cwt. of rice	,,	26*s*. per cwt.	,,	4*d*. ,,
10 lbs. of cocoa	,,	1*s*. 1*d*. per lb.	,,	1*s*. 4*d*. ,,
15 ,, ,,	,,	$6\frac{3}{4}$*d*. ,,	,,	8*d*. ,,
Sundries		£19. 12*s*.		£25. 8*s*. 6*d*.

Find (*a*) the weekly profit realized; (*b*) the gain or loss % on each article; (*c*) the total gain %.

(15) If an investment of £91. 17*s*. 6*d*. yield me £3. 10*s*. per annum, what percentage do I get?

§ 4. Converse of Interest.

$$\text{Since} \quad I = \frac{p \times r \times t}{100},$$

$$100 \times I = p \times r \times t,$$

$$\text{and} \quad \frac{100 \times I}{p \times r} = t \qquad \text{(V.)}$$

$$\frac{100 \times I}{p \times t} = r \qquad \text{(VI.)}$$

$$\frac{100 \times I}{r \times t} = p \qquad \text{(VII.)}$$

If, then, of the four quantities, p, r, t, I, any three be given, the fourth can be found from the formulæ (II.), (V.), (VI.), (VII.).

Find the interest on £540 at 6 % for $4\frac{1}{2}$ months.

$$I = \frac{p \times r \times t}{100} = \frac{540 \times 6 \times \cdot 375}{100} = 5\cdot 4 \times 6 \times \cdot 375 = 12\cdot 15. \qquad \textit{Ans.}\ £12.\ 3s.$$

In what time will £540 at 6 % yield an interest of £12. 3*s*.?

$$t = \frac{100 \times I}{p \times r} = \frac{100 \times 12\cdot 15}{540 \times 6} = \frac{121\cdot 5}{324} = \cdot 375 = \tfrac{375}{1000}\big|\tfrac{3}{8}.$$

Ans. $\frac{3}{8}$ of a year, or $4\frac{1}{2}$ months.

At what rate will £540 in $4\frac{1}{2}$ months yield an interest of £12. 3*s.*?

$$r=\frac{100\times I}{p\times t}=\frac{100\times 12{\cdot}15}{540\times {\cdot}375}=\frac{1215}{54\times 3{\cdot}75}=6. \qquad \textit{Ans. } 6\ \%.$$

What principal will in $4\frac{1}{2}$ months at 6 % yield an interest of £12. 3*s.*?

$$p=\frac{100\times I}{r\times t}=\frac{100\times 12{\cdot}15}{6\times {\cdot}375}=\frac{1215}{2{\cdot}25}=540. \qquad \textit{Ans. } £540.$$

In what time will the interest on £318. 14*s.* 3*d.* at £4. 12*s.* % amount to £16. 4*s.* 9*d.*?

$t=\dfrac{100\times I}{p\times r}=\dfrac{100\times 16{\cdot}2375}{318{\cdot}7125\times 4{\cdot}6}=\dfrac{1623{\cdot}75}{318{\cdot}7125\times 4{\cdot}6}$; multiplying both terms by 8* $=\dfrac{12990}{2549{\cdot}7\times 4{\cdot}6}$.

$$\begin{array}{rr}
25497 & \\
46 & \\
\hline
152982 & \\
101988 & \\
\hline
1172862)1299000(1{\cdot}10755 \\
126138 \quad 6 & \\
8852 & \\
642 & \\
56 &
\end{array}
\qquad
\begin{array}{r}
{\cdot}1076 \\
563 \\
\hline
323 \\
64 \\
5 \\
\hline
39{\cdot}2
\end{array}
\left.\begin{array}{l}\\ \\ \end{array}\right\}\text{keeping one place.}$$

Ans. 1 year, 39 days.

Exercise L.

(1) In what time will the interest on £455. 10*s.* amount to £1. 17*s.* $11\frac{1}{2}$*d.* at 5 %?

(2) In what time will the interest on £368. 15*s.* 4*d.* amount to £15 at £6. 13*s.* 10*d.* %?

(3) In what time will a principal of £360 amount† to 400 guineas at 5 %?

(4) At what rate will the interest on £15 in $4\frac{1}{2}$ months amount to 7*s.* $10\frac{1}{2}$*d.*?

(5) What principal will, at £4. 18*s.* 9*d.* %, yield 500 guineas a year?

(6) In 2 years, 53 days, a principal of £1000 amounted to £1200. Find the rate charged.

* See foot-note, p. 175. † See foot-note, p. 179.

(7) What sum will, at $8\frac{3}{4}$ %, yield 19 guineas interest in 3 months?

(8) Find x in each of the following:

	Principal.	Rate %.	Time.	Interest.
a.	£465,	$3\frac{1}{2}$,	47 days,	x.
b.	£x,	£4. 17*s.*,	1 yr., 4 mos.,	£70. 14*s.* 7*d.*
c.	£24,000,	x,	1 year,	£900.
d.	£230. 17*s.* 3*d.*,	$2\frac{1}{2}$,	x,	£3. 8*s.* 8*d.*

§ 5. DISCOUNT.

What sum of money will, at $3\frac{1}{2}$ %, in $2\frac{1}{2}$ months *amount to £75*?

The formula, $p = \frac{100 \times I}{r \times t}$, will not avail us here, as I is unknown. We have recourse, therefore, to an artifice which is known in old books by the name of "Supposition," subsequently worn down to "Position," and which is applicable to a large class of questions besides the present.

Assume or *suppose* any sum of money whatever, say £86, laid out for the given time at the given rate, and find the interest thereon, and hence the amount.

$$\text{Interest on £86} = \frac{86 \times 3{\cdot}5 \times 2{\cdot}5}{100 \times 12} = \text{£}{\cdot}62708\dot{3}.$$

$$\text{Amount} \quad = \text{£}86{\cdot}62708\dot{3}.$$

We have now the following question in Proportion:

$$\text{If £86 amounts to £}86{\cdot}62708\dot{3},$$
$$\text{£}x \quad \text{,,} \quad \text{£}75.$$

$$86{\cdot}62708\dot{3} : 75 = 86 : x.$$

$$86{\cdot}627\cancel{08\dot{3}})\,64500000\,(74{\cdot}457$$

```
3861042
 395959
  49451
   6137
```

Ans. £74. 9*s.* $1\frac{3}{4}$*d.*

Since for the "position" we may choose any sum we please, it is best to select the easiest, which obviously is in this case £100.

$$\text{Interest on £100} = \frac{\cancel{100} \times 3{\cdot}5 \times 2{\cdot}5}{\cancel{100} \times 12} = {\cdot}7291\dot{6}.$$

$$\text{If, then, £100 amounts to £}100{\cdot}7291\dot{6},$$
$$\text{£}x \quad \text{,,} \quad \text{£}75.$$

$$100{\cdot}7291\dot{6} : 75 = 100 : x.$$

```
1·007291)75·0000(74·457
           4 4896
            4604
             575
              71
```

Ans. £74. 9*s.* 1¾*d.* as before.

Hence, generally:

What sum of money will, at r %, in t years amount to £A?

Interest on £100 $= r \times t$,
Amount ,, $= 100 + r \times t$.

Hence the proportion:

$$100 + r \times t : A = 100 : x,$$

and $$x = \frac{100 \times A}{100 + r \times t}.$$

The sum of money which at the given rate and time will amount to A is called the present value of A, which we will call V. Hence:

$$V = \frac{100 \times A}{100 + r \times t} \qquad \text{(VIII.)}$$

The sum of money paid in cash for the transfer of a debt due some time hence, should be the present value thus found. The difference between this present value and the amount of the debt is called Discount, which we will call D. Hence:

$$D = A - V \qquad \text{(IX.)}$$

D can, however, be obtained independently, thus:

If $r \times t$ is the discount on $100 + r \times t$,
x ,, A.

$$100 + r \times t : A = r \times t : x,$$

$$\therefore D = \frac{A \times r \times t}{100 + r \times t} \qquad \text{(X.)}$$

Discount questions occur most frequently in connection with Bills of Exchange.

Find the *true* discount on a bill for £387. 14*s.* 10*d.*, dated May 5th, at 3 months, and discounted June 4th, interest being calculated at 4 %.

This means that £387. 14*s.* 10*d.* will fall due on August 5th, but in most countries the usage of commerce allows 3 days' "grace," and accordingly the money will be payable August 8th. From June

4th to August 8th are 65 days, which is the time the bill has yet to run.

$$\therefore\ D=\frac{£387.\ 14s.\ 10d.\times 4\times\frac{65}{365}}{100+4\times\frac{65}{365}}.$$

$$r\times t=4\times\frac{\overset{13}{\cancel{65}}}{\underset{73}{\cancel{365}}}=\frac{52}{73}=\cdot\dot{7}123287\dot{6}$$

$$\begin{array}{l} 73)\ 520\ (\cdot\dot{7}1232'8'7'\dot{6}' \\ \quad 90 \\ \quad 170 \\ \quad\ 240 \\ \quad\quad 21 \\ \quad\quad \&c. \end{array}$$

$$D=\frac{387\cdot741\dot{6}\times\cdot\dot{7}123287\dot{6}}{100\cdot\dot{7}123287\dot{6}}$$

$$\begin{array}{r} 387\cdot742 \\ 9232\ 17 \\ \hline 271419 \\ 3877 \\ 775 \\ 116 \\ 8 \\ 3 \\ \hline \end{array}$$

Or see ring G, p. 150.

$$\begin{array}{l} 100\cdot\cancel{7}\cancel{1}\cancel{2}\cancel{3})\ 276\cdot198\ (2\cdot742 \\ \qquad\qquad\quad 74\ 773 \\ \qquad\qquad\quad\ 4\ 275 \\ \qquad\qquad\qquad 247 \end{array}$$

Ans. £2. 14*s.* 10¼*d.*

Practically, in England, the true discount is never calculated, but the banker deducts simple interest from the amount of the bill. Comparing the formulæ for I and D, we find the numerators the same, while the denominator in the former is 100, and in the latter $100+r\times t$.

In the above sum the interest is:

$$\frac{387\cdot741\dot{6}\times\cdot\dot{7}123287\dot{6}}{100}=2\cdot76198=£2.\ 15s.\ 2\tfrac{3}{4}d.$$

Deducting the true discount we get $4\frac{1}{2}d.$, which is the banker's surcharge.

EXERCISE LI.

(1) Find the present value of:

a. £427. 10*s.* 6*d.*,	discounted at	$3\frac{1}{2}$ %,	due in	3	months.
b. £6359. 18*s.* 4*d.*	„	$4\frac{3}{4}$ %,	„	·4	„
c. £794. 11*s.*	„	5 %,	„	6	„
d. £82. 13*s.* 4*d.*	„	$5\frac{3}{8}$ %,	„	70	days.

e. £445. 10*s.*, discounted at $3\frac{7}{8}$ %, due in 94 days.

f. £1250 „ £2. 17*s.* 8*d.* %, due in 114 days.

(2) Find the discount on a bill of:

a. £700, due March 5—8, discounted January 4, at 5 %.

b. £376, due August 20—23, discounted May 11, at $4\frac{1}{2}$ %.

c. £40. 10*s.* 6*d.*, due September 30—October 3, discounted September 17, at $3\frac{1}{4}$ %.

d. £461. 3*s.* 6*d.*, due October 9—12, discounted May 5, at $7\frac{1}{2}$ %.

(3) Find the present value of a bill of:

a. £629. 12*s.*, drawn May 11, at 3 months, discounted June 2, at $5\frac{3}{8}$ %.

b. £485. 19*s.* 3*d.*, drawn July 31,* at 4 months, discounted November 11, at $8\frac{1}{2}$ %.

c. £374. 16*s.*, drawn March 10, at 90 days, discounted April 1, at $6\frac{7}{8}$ %.

(4) Find the difference between simple interest and discount on:

a. £760 at $3\frac{5}{8}$ % for 68 days.

b. £1848. 10*s.* 10*d.* at $4\frac{2}{5}$ % for 80 days.

c. £2466. 13*s.* 4*d.* at $5\frac{3}{5}$ % for 99 days.

In a similar manner may be treated questions such as the following:

For how much should goods worth £540 be insured at 3 %, so as to recover in case of loss both goods and premium?

Goods worth £97	↓	should be insured for	↓	£100.
„ £540	↓	„ „	↓	£x.

$$97 : 540 = 100 : x.$$

```
97) 54000 (556·701
     550
      650
       680
        100
```

Ans. £556. 14*s.*

If goods sold at a profit of $12\frac{1}{2}$ % realized £360, find the cost price.

We have only to remember that gain or loss is always calculated on the cost price.

* This will be due November 30—December 3.

Cost price, £100, ↓ Selling price, £112·5. ↓
x £360.

$$112{\cdot}5 : \cancel{360} = 100 : x$$
$$\cancel{450} \quad 4 \quad \cancel{400}$$
$$\cancel{5} \qquad 80$$
$$4 \times 80 = 320$$

Ans. £320.

Exercise LII.

(1) If by selling an article at £1. 1*s.* 9*d.* a pound I gain 16 %, what was the prime cost?

(2) I bought 100 gallons of brandy at 17*s.* 6*d.* per gallon; $9\frac{1}{2}$ gallons are lost by leakage; the remainder is put into bottles holding $1\frac{1}{2}$ pints; at what price per bottle must it be sold so as to gain $18\frac{1}{2}$%?

(3) For how much should goods worth £635 be insured at $2\frac{1}{2}$ % to recover the value of goods and premium?

(4) Also goods worth £100 at 5 %?

(5) " £67. 10*s.* at 4 %?

(6) Goods bought at £2. 5*s.* 10*d.* per cwt. are sold at £2. 14*s.* 1*d.*; what is the gain per cent.?

(7) Of goods worth £1000, one-third is sold at a profit of 15 %; for how much must the remainder be sold to gain 20 % on the whole?

(8) I bought sugar at £1. 18*s.* 6*d.* per cwt.; at how much per lb. must I sell it to lose 4 %? (To be worked accurately.)

(9) If by selling wine at 15*s.* a gallon I lose 10 %, at what price must I sell it to gain 15 %?

(10) A person buys coffee at £5. 12*s.* 6*d.* per cwt, and chicory at £2. 5*s.* 5*d.*, and mixes them in the proportion of 2 of chicory to 5 of coffee. He retails the mixture at 1*s.* 3*d.* per lb.; what is his gain or loss %?

(11) I bought 4 cwt. of sugar at 4*d.* per lb., and 7 cwt. at 5*d.* per lb.; at how much per lb. must I sell the mixture to gain 20 %?

(12) A bought 150 eggs at 2 a penny, and 150 at 3 a penny; he sold them all at 5 for 2*d.* How much % did he gain or lose?

(13) I bought 580 metres of silk at 6·65 fr. per metre, and sold 300 yards at 6*s.* $9\frac{1}{2}$*d.* per yard, and the remainder at 7*s.* per yard;

find the gain or loss %, reckoning 1 metre = 39·37 in., and £1 = 25·22 fr.

(14) If apples are bought at 4 for $1\frac{1}{2}d.$, how many should be sold for $3\frac{1}{2}d.$ to gain 75 % ?

(15) I bought a Geneva watch and paid duty at the rate of 10 %. I sold the watch for £21, making a profit of 15 % on my whole outlay. Find the original cost in francs at $9\frac{1}{2}d.$ each.

§ 6. STOCKS AND INVESTMENTS.

Stock is another word for capital, not expressly divided into shares, in any joint undertaking, whether that undertaking be the loan of money to a government or the placing it in the hands of the managers of a trading company.

From various causes the market value of this stock fluctuates. When the market price is the same as the original money laid down, the stock or shares are said to be at par; when less, at a discount; when more, at a premium. We must therefore carefully distinguish between two kinds of principal, viz., *stock* money or *sterling* value, the former being the original price, or at any rate the sum credited to the holder's name in the books of the company or government; the latter, the market value per cent.

On June 22, 1870, Consols (i.e. stock in the Consolidated Three per Cent. Annuities of England) are quoted at $92\frac{5}{8}$, which means that a claim on the government for the yearly interest on £100 can be bought in the market for £92. 12*s.* 6*d.*, or that £100 stock = £92. 12*s.* 6*d.* sterling.

This leads to such questions as the following:

a. How much stock for a given sum in sterling money ?

b. How much sterling money for a given quantity of stock ?

How much stock can be bought for £2416. 10*s.* at $92\frac{5}{8}$? *

x stock	£2416·5 sterling
$92\frac{5}{8}$ sterling	100 stock

```
92625) 241650000 (2608·906
        564000
          825000
           84000
             637
              81
```

Ans. £2608. 18*s.* $1\frac{1}{2}d.$

* All problems worked by chain rule can also be worked by direct proportion, and *vice versa* (see p. 79).

How much money will be realized by selling £815 Consols at $92\frac{5}{8}$?

$$\begin{array}{r|l} x \text{ sterling} & 815 \text{ stock} \\ 100 \text{ stock} & 92\frac{5}{8} \text{ sterling} \end{array}$$

$$\begin{array}{r} \cdot 92625 \\ 518 \\ \hline 741000 \\ 9263 \\ 4631 \\ \hline 754 \cdot 894 \end{array}$$

Ans. £754. 17*s.* $10\frac{3}{4}$*d.*

Exercise LIII.

(1) How much stock can be bought for £8450 at $93\frac{1}{4}$?

(2) " " £748. 16*s.* at $89\frac{7}{8}$?

(3) What will be realized by selling out £350 Consols at $91\frac{1}{2}$?

(4) " " £68. 4*s.* 6*d.* Consols at $87\frac{3}{4}$?

§ 7. As a matter of practice, every transfer of shares is effected by a broker, who charges $\frac{1}{8}$ % to both buyer and seller. The purchaser has thus to pay $\frac{1}{8}$ % more, and the vendor receives $\frac{1}{8}$ % less, than the price quoted.

How much stock can be bought for £520 at $93\frac{1}{2}$, with $\frac{1}{8}$ brokerage?

Each £100 stock costs $93\frac{1}{2} + \frac{1}{8} = £93\frac{5}{8}$ sterling.

$$\begin{array}{r|l} x & 520 \\ \cancel{93\frac{5}{8}} & 100 \\ 749 & 8 \end{array}$$

$$\begin{array}{l} 74\cancel{9})\ 416000\ (555 \cdot 407 \\ \quad\ \ 4150 \\ \quad\quad 4050 \\ \quad\quad\ \ 3050 \\ \quad\quad\quad\ \ 54 \end{array}$$

Ans. £555. 8*s.* $1\frac{3}{4}$*d.*

How much will be realized by selling out £520 stock at $89\frac{5}{8}$; brokerage $\frac{1}{8}$ %?

Each £100 stock realizes $89\frac{5}{8} - \frac{1}{8} = £89\frac{1}{2}$.

$$\begin{array}{r|l} x & 52\cancel{0} \\ 10\cancel{0} & 89 \cdot 5 \end{array}$$

$8 \cdot 95 \times 52 = 465 \cdot 4.$ *Ans.* £465. 8*s.*

EXERCISE LIV.

Find the sum to be paid for:

(1) £8350 3 per cent. Consols, at $94\frac{1}{2}$, brokerage $\frac{1}{8}$.

(2) £746. 18*s*. 6*d*. 3 per cent. Consols, at $91\frac{7}{8}$, brokerage $\frac{1}{8}$.

(3) £844. 4*s*. 4*d*. 3 per cent. Consols, at $89\frac{1}{4}$, brokerage $\frac{1}{8}$.

(4) £768. 17*s*. 11*d*. 3 per cent. Consols, at 88, brokerage $\frac{1}{8}$.

What will be realized by selling out:

(5) £600 3 per cent. Consols, at 90, brokerage $\frac{1}{8}$.

(6) £937. 12*s*. 11*d*. 3 per cent. Consols, at $90\frac{1}{2}$, brokerage $\frac{1}{8}$.

(7) £666. 13*s*. 4*d*. 3 per cent. Consols, at $86\frac{5}{8}$, brokerage $\frac{1}{8}$.

(8) £27468. 10*s*. 3 per cent. Consols, at $91\frac{1}{4}$, brokerage $\frac{1}{8}$.

§ 8. The operations of most frequent occurrence on the stock exchange lead to questions like the following:

Bought £16,000 Consols at $93\frac{5}{8}$, and sold out at $94\frac{1}{8}$. Find profit, allowing for brokerage $\frac{1}{8}$ % on each operation.

$$\text{Profit on £100 stock} = (94\tfrac{1}{8} - \tfrac{1}{8}) - (93\tfrac{5}{8} + \tfrac{1}{8}) = 94 - 93\tfrac{3}{4} = \tfrac{1}{4},$$
$$\text{,, \quad £16,000} \quad = 160 \times \tfrac{1}{4} = £40.$$

Bought fr. 25,000 in the French Rentes at 71·75, and sold at 66·90. Find the loss (brokerage $\frac{1}{8}$ % both ways).

$$\text{Loss on 100 fr. stock} = (71{\cdot}75 + {\cdot}125) - (66{\cdot}90 - {\cdot}125) = 5{\cdot}1,$$
$$\text{,, \quad fr. } 25{,}000 = 250 \times 5{\cdot}1 = 1275 \text{ fr.}$$

EXERCISE LV.

(1) If the English 3 per cent. funded debt amounts to £723,120,000, find the diminution of its value in the market caused by the outbreak of the German-French war in 1870. Quoted $92\frac{5}{8}$ on June 22, and $88\frac{3}{8}$ August 5.

(2) Find the profit or loss on each of the following operations (brokerage $\frac{1}{8}$ % on each transaction):

	Stock.	Bought in at.	Sold out at.
a.	£80,000,	$91\frac{5}{8}$,	90.
b.	£17,500,	$88\frac{7}{8}$,	$89\frac{1}{8}$.
c.	£51,600,	$86\frac{1}{4}$,	$89\frac{1}{2}$.
d.	fr. 30,000,	65·75,	67·35.

§ 9. What yearly income will be derived from the investment of £7850 in the 3 per cents. at 90?

£90 sterling buys an annuity of £3.

x	$785\cancel{0}$	$3)\underline{785\ \ 0\ \ 0}$
$\cancel{9}\cancel{0}$	$\cancel{3}$	£261. 13*s*. 4*d*.
3		

Ans. £261. 13*s*. 4*d*.

How much must I invest in the 3 per cents. at $90\frac{5}{8}$ to obtain an income of £150 a year (brokerage $\frac{1}{8}$ %)?

Each £100 share costs $90\frac{5}{8} + \frac{1}{8} = 90\frac{3}{4}$.

x	£$\cancel{1}\cancel{5}\cancel{0}$	
£$\cancel{3}$	90·75	$90·75 \times 50 = 4537·5$
	50	

Ans. £4537. 10*s*.

If I invest in the 4 per cents. at $89\frac{1}{2}$, what rate of interest do I get for my money?

x	100	89$\cancel{5}$) 4000 (4·469
89·5	4	4200
		620
		83
		2

Ans. £4. 9*s*. $4\frac{3}{4}$*d*. %.

EXERCISE LVI.

(1) What yearly income will be derived from £895 invested in the 5 per cents. at 105?

(2) What yearly income is derived from the investment of £10,000 in the 3 per cent. Consols at $91\frac{1}{2}$?

(3) Find the yearly income derived from investing £750 in the 5 per cent. Austrian Metalliques at $48\frac{7}{8}$.

(4) How much must I invest in the 4 per cents. at $93\frac{5}{8}$ to realize a yearly income of £120?

(5) How much must be invested in the $3\frac{1}{2}$ per cents. at $87\frac{3}{4}$ to realize a six-monthly dividend of £75?

(6) Work these five sums, allowing $\frac{1}{8}$ % brokerage.

(7) Find the yearly rate of interest obtained from the following investments:

a. Consols 3 per cent. at $88\frac{3}{8}$.

b. French Rentes 3 per cent. at 66·90.

c. Prussian 5 per cent. at 88.

d. Russian $4\frac{1}{2}$ per cent. at 86.
e. Turkish 6 per cent. (1854) at $81\frac{1}{2}$.
f. Turkish 5 per cent. (1865) at $38\frac{1}{2}$.

§ 10. Which is the most profitable stock for investment, the 4 % at 85, or the 3 % at 63?

Suppose £100 invested in each kind, the 4 % will yield $\frac{100 \times 4}{85} = 4{\cdot}706$; the 3 % will yield $\frac{100 \times 3}{63} = 4{\cdot}762$. *Ans.* The 3 %.

It is simpler to suppose £1 invested in each kind, and then the resulting fractions are $\frac{4}{85}$ and $\frac{3}{63}$, which decimalized become ·04706 and ·04762. $\frac{3}{63}$, then, is the larger fraction, and the 3 % are the better investment. *Ans.* The 3 %.

§ 11. Find the difference in income between investing £866. 13*s.* 4*d.* in the 5 per cents. at $107\frac{5}{8}$ and in the $4\frac{1}{2}$ per cents. at $95\frac{1}{4}$, brokerage $\frac{1}{8}$.

£1 in the 5 per cents. yields $\frac{5}{107\frac{3}{4}} = £{\cdot}0464037$

£1 „ $4\frac{1}{2}$ „ $\frac{4\frac{1}{2}}{95\frac{3}{8}} = £{\cdot}0471821$

Difference in income on £1 $= {\cdot}0007784$

„ „ £$866{\cdot}\dot{6}$ $= 866{\cdot}\dot{6} \times {\cdot}0007784$.

$$\begin{array}{r} {\cdot}0007784 \\ 768 \\ \hline 622 \\ 46 \\ 5 \\ \hline {\cdot}673 \end{array}$$

Ans. 13*s.* $5\frac{3}{4}$*d.*

Exercise LVII.

(1) Which of the following pairs is the more profitable investment:

a. 4 per cents. at 95, and $3\frac{1}{2}$ per cents. at 84.
b. $3\frac{1}{2}$ „ $67\frac{1}{4}$, and 4 „ $81\frac{1}{2}$.
c. $5\frac{3}{4}$ „ $110\frac{1}{2}$, and 3 „ $67\frac{5}{8}$.

(2) Find the differences in income, supposing £584. 10*s.* were invested in each of the pairs in (1).

(3) Find the difference in income between investing £1500 in the 3 per cent. Consols at $88\frac{7}{8}$, and in $12\frac{1}{2}$ per cent. Bank stock at $185\frac{1}{2}$, brokerage $\frac{1}{8}$.

§ 12. What should be the price of the 3 per cents. in order that investments in them should make 5 % interest?

$$\begin{array}{r|l} x & 3 \\ \cancel{5} & \cancel{100} \\ & 20 \end{array} \qquad 3 \times 20 = 60. \qquad \textit{Ans. } 60.$$

What should be the price of the $3\frac{1}{2}$ per cents. to yield the same rate of interest as the 4 per cents. at 95?

$$\begin{array}{r|l} x & 3{\cdot}5 \\ 4 & 95 \end{array} \qquad \frac{3{\cdot}5 \times 95}{4} = 83\tfrac{1}{8}. \qquad \textit{Ans. } 83\tfrac{1}{8}$$

Exercise LVIII.

(1) Find the price of $4\frac{1}{2}$ % to equal the $3\frac{1}{2}$ % at $88\frac{1}{2}$.
(2) „ $3\frac{3}{4}$ „ 4 „ $92\frac{1}{4}$.
(3) „ 4 „ 5 „ $100\frac{7}{8}$.
(4) „ $2\frac{1}{2}$ „ $3\frac{1}{2}$ „ $68\frac{5}{8}$.
(5) What should the 5 % be when the 3 % are at $89\frac{1}{2}$?
(6) „ $3\frac{1}{2}$ „ $5\frac{3}{4}$ „ par?

§ 13. If I transfer £3850 from the $3\frac{1}{2}$ per cents. at $91\frac{3}{8}$ to the 5 per cents at $102\frac{1}{2}$, find the alteration in (*a*) the amount of stock held, (*b*) the yearly income, (*c*) the rate of interest for my money.

a.

$$\begin{array}{r|l} x^{*} & 3850 \\ 1\cancel{00} & 91{\cdot}375 \\ 102{\cdot}5 & 1\cancel{00} \end{array} \qquad \frac{3850 \times 91{\cdot}375}{102{\cdot}5} = 3432{\cdot}134.$$

3850
3432·134
Difference 417·866 *Ans.* £417. 17*s.* 4*d.* decrease.

b.

$$\begin{array}{r|l} x & 3850 \\ 100 & 91{\cdot}375 \\ 102{\cdot}5 & 5 \end{array} \qquad\qquad \begin{array}{r|l} x & 3850 \\ 100 & 3{\cdot}5 \end{array}$$

Ans. 171·607. *Ans.* 134·75.

171·607 income in new investment.
134·75 „ old „
Difference 36·857 *Ans.* £36. 17*s.* $1\frac{3}{4}$*d.* increase.

* Read, How much 5 per cent. stock for £3850 $3\frac{1}{2}$ per cent. stock, if £100 $3\frac{1}{2}$ per cent. stock yields £91·375 sterling, and £102·5 sterling buys £100 5 per cent. stock?

c.

$$\begin{array}{r|l} x & 100 \\ 91\frac{3}{8} & 3\frac{1}{2} \end{array} \qquad \begin{array}{r|l} x & 100 \\ 102\cdot5 & 5 \end{array}$$

Ans. 3·830. *Ans.* 4·878.

	4·878	rate of interest in new investment.	
	3·830	„ old „	
Difference	1·048	*Ans.* £1. 0*s.* 11½*d.* % increase.	

EXERCISE LIX.

Find the alteration in the amount of stock held, the income derived and the rate %, if the following transfers are effected:

	Amount transferred	from	at	to	at.
(1)	£7850,	$3\frac{1}{2}$ %,	66·90,	5 %,	90.
(2)	£5465,	$4\frac{3}{4}$ „	$90\frac{1}{2}$,	$2\frac{1}{2}$ „	$47\frac{1}{2}$,
(3)	£690,	3 „	$76\frac{1}{2}$,	4 „	$82\frac{5}{8}$,
(4)	£3567. 14*s.* 10*d.*,	3 „	88,	$3\frac{1}{2}$ „	par.
(5)	£1487. 11*s.* 8*d.*,	$3\frac{1}{4}$ „	90,	$6\frac{1}{2}$ „	180.
(6)	10,000 guineas,	3 % Consols,	$88\frac{3}{8}$,	5 % Italians at $46\frac{1}{2}$.	

§ 14. SHARES. Calculations on stock are, with slight alterations, applicable also to shares, and it is most convenient preliminarily to suppose share capital converted into stock, as this enables us to calculate by percentages, irrespective of the issue value of the shares. Thus shares issued at £20, "fully paid up," and sold at a discount of £6, i.e. for £14, may be compared to £100 stock at £70, share and price being both multiplied by 5.

If shares in a certain company, issued at 25 and bought at 2 premium, yield a six-monthly dividend of £1. 16*s.*, find (*a*) the sterling value of the shares per cent., (*b*) the rate of interest, and (*c*) the yearly income derived from investing £540 in these shares.

a. $\begin{array}{r|l} x & 100 \\ 25 & 27 \end{array}$ *Ans.* £108.

b. $\begin{array}{r|l} x & 100 \\ 27 & 2\times1\cdot8 \end{array}$ *Ans.* £13⅓ %.

c. $\begin{array}{r|l} x & 540 \\ 27 & 2\times1\cdot8 \end{array}$ *Ans.* £72.

The issue price of shares is usually not "paid up" on application, but "called in" by the managers according to necessity or stipulation.

Find the rate of interest on an investment in railway shares issued at £50, on which were paid £10 on application and four successive

calls of £5 each, bought at $1\frac{5}{8}$ premium, and yielding a six-monthly dividend of 17*s*. 6*d*. per share.

Capital invested : £10 + 4 × £5 + £$1\frac{5}{8}$ = £31. 12*s*. 6*d*.

x	100	253) 1400 (5·533	
31·625	·875 × 2	1350	
		850	
		91	*Ans.* £5. 10*s*. 8*d*. %.

Exercise LX.

(1) Find the six-monthly dividend derived from investing 1000 guineas in £50 railway shares at par, yielding $3\frac{1}{2}$ per cent. per annum.

(2) Find the price per cent. of mining shares issued at £15 and sold at $2\frac{1}{2}$ discount.

(3) Find the rate of interest on shares, £35 paid up, yielding a half-yearly dividend of £2. 14*s*.

(4) Find the rates of interest obtained by investing in the following railway shares :

a. London and North Western,	quoted at $124\frac{1}{2}$,	paying	6 % per an.
b. Midland,	„ 124,	„	$6\frac{1}{4}$ % „
c. Great Western,	„ $67\frac{3}{4}$,	„	$3\frac{9}{40}$ % „
d. South Eastern,	„ 67,	„	$2\frac{1}{2}$ % „

(5) Find the yearly income derived from the investment of every £100 in the Austrian 5 per cents. at 48·55, deducting 16 % income tax.

(6) I invested £1450 in the 3 per cents. at $88\frac{1}{2}$, and sold out when they had risen $2\frac{1}{4}$ %. What was my gain ?

(7) How much stock must I sell out of the $3\frac{1}{2}$ per cents. at $81\frac{7}{8}$ to enable me to buy £5000 4 per cent. stock at $94\frac{1}{2}$, brokerage $\frac{1}{8}$ in each transaction ?

(8) What is the price of stock if £8000 stock can be bought for £5830 ?

(9) My half-yearly dividend from the 3 per cents. is £247. 10*s*. How much stock do I hold, and what shall I realize by selling out at $89\frac{1}{2}$, brokerage $\frac{1}{8}$?

(10) How much stock must I sell out from the 4 per cents. at $96\frac{7}{8}$ to raise a sufficient sum for discounting a bill for £1000, due 52 days hence and discounted at $5\frac{1}{2}$ % ?

(11) Find the yearly income derived from investing a legacy of £4583. 10*s*. in the 3 per cent. Consols at $91\frac{7}{8}$, allowing for legacy duty 5 %, brokerage $\frac{1}{8}$ %, deducting an income tax of 5*d*. in the £.

(12) I invested £1460 in the $4\frac{1}{2}$ per cents. at $100\frac{1}{4}$, and sold when they had fallen, losing £100, inclusive of the double brokerage. Find the selling price.

(13) What would this selling price have been if I had cleared £100?

(14) I bought Jan. 1st, £5000 Consols at $92\frac{3}{8}$; I sold Feb. 10th, £1500 of this stock at $93\frac{7}{8}$, March 12th, £2000 at 93, and the remainder April 1, at $92\frac{5}{8}$, brokerage $\frac{1}{8}$ % on each transfer. Find my total gain or loss, supposing that I could have made 5 % per annum by other investments.

(15) I invested £2460 in the 3 per cents. at $90\frac{3}{8}$, with $\frac{1}{8}$ % brokerage; on their falling to $87\frac{1}{2}$, I sold out, paying again $\frac{1}{8}$ % brokerage, and put my money out on a mortgage at $4\frac{1}{2}$ %. Find the alteration in my capital and in my yearly income.

(16) I bought £5000 stock at $88\frac{1}{2}$. At what price must I sell it to gain £100?

(17) If I buy at 85, at what price must I sell to make $12\frac{1}{4}$ % profit, brokerage $\frac{1}{8}$ on each transaction.

(18) I invested money in the 3 per cents. at $75\frac{3}{8}$, and after drawing half a year's dividend, I sold out at a rise of $1\frac{7}{8}$ %, increasing my capital altogether by £91. 2*s.* 6*d.* How much did I originally invest?

(19) I invested in the 3 per cents. at $89\frac{1}{2}$, brokerage $\frac{1}{8}$. How much stock did I purchase, and what was the broker's commission, if I paid for investment and commission £410?

(20) The issue price of certain railway shares was £50, to be paid in five instalments of £10 each, the first payment on application. After a "call" or second payment of £10, the shares stood at £1 per share premium. I then invested £756, and after paying a further call of £10, a dividend was declared of $8\frac{1}{2}$ % per annum. on the paid-up capital. What is the amount of my dividend, and what interest do I get for my money?

(21) A friend lent me £475 at 4 %; but to raise the money he sold out 3 per cent. Consols when they were at $87\frac{5}{8}$ (brokerage $\frac{1}{8}$). I kept the money three months, and meanwhile the funds rose to $91\frac{1}{2}$. How much have I to pay my friend to cover the interest and to replace the stock he previously held (brokerage again $\frac{1}{8}$)?

CHAPTER XI.

SQUARE AND CUBE ROOT.

§ 1. If a number be multiplied by itself, the product is called the SQUARE of that number, and the number is called the SQUARE ROOT of the product.

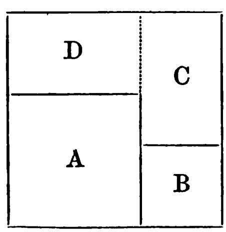
Roots:	1,	2,	3,	4,	5,	6,	7,	8,	9,	&c.
Squares:	1,	4,	9,	16,	25,	36,	49,	64,	81,	&c.

If this table were carried out ad infinitum, inspection would shew us the root of any proposed square. In absence of such a table, methods must be found to answer the same purpose.

§ 2. $1^2 = 1$; $10^2 = 100$; $100^2 = 10000$; $1000^2 = 1000000$, &c.; from which we see that a power of ten is squared by doubling its ciphers. Again, any number between 100 and 1000 will have a square between 10000 and 1000000, and will therefore have either five or six figures; or, generally:

Numbers with	1 figure	have in their squares	1 or 2	figures.
„	2 figures	„	3 or 4	„
„	3	„	5 or 6	„
„	4	„	7 or 8	„
	⋮		⋮ ⋮	
„	n	„	$2 \times n - 1$, or $2 \times n$	figures.

§ 3. The expression, "the square" of a *number*, is derived from the fact that to find the area of a square *surface* we multiply the number of units of length in one side by itself, the result being square units.



Suppose the whole side of a square of 8 units to be divided into

two parts of 5 and 3 units respectively, then the whole square will have 64 square units, while the sum of the two smaller squares, A and B, will be $25+9$, or 34 square units. This differs from the whole by 30 square units, which is consequently the area of the two surfaces C and D; and as each of these surfaces is 5 units long and 3 units broad, and therefore has 15 square units, we see that this is correct.

§ 4. Square the number 37.

$$\begin{aligned} 37 \times 37 &= (30 \times 37) + (7 \times 37) \\ &= (30 \times 30) + (30 \times 7) + (7 \times 30) + (7 \times 7) \\ &= (30 \times 30) + 2 \times (30 \times 7) + 7 \times 7; \end{aligned}$$

$$\text{or } 37^2 = 30^2 + 2 \times 30 \times 7 + 7^2;$$

$$1369 = 900 + 420 + 49.$$

Similarly, $8^2 = 5^2 + 2 \times 5 \times 3 + 3^2$, as in the illustration of § 3; and, generally, if $A = a + b$, $A^2 = a^2 + 2 \times a \times b + b^2$.

Learn by heart: *The square of the sum of two numbers = the sum of their squares + twice their product.*

§ 5. We may also notice that if one of the two numbers be the greater, the product of the two numbers is greater than the square of the less, and still more is twice the product greater than the square of the less.

§ 6. Find the square root of 4096. From § 2, we find that as the square has 4 figures, the root has 2 figures, or, which is the same thing, the root is between 10 and 100.

The square of	10	is	100
"	20	"	400
"	30	"	900
"	40	"	1600
"	50	"	2500
"	60	"	3600
"	70	"	4900

whence it appears that the root lies between 60 and 70.

Subtract, then, 60^2 from 4096; remainder, 496.

$$\begin{array}{lr} & 4096 \\ \text{A} & \underline{3600} \\ & 496 \end{array}$$

This remainder, now, contains not only the square of the part of the root yet wanting, but also 2×60 (or 120) times that part; and as this last is much the larger of the two quantities (§ 5), we may, to begin with, disregard the square of the number sought, and try to find that number itself by dividing 496 by 120.

$$\begin{array}{r} 120)\,496\,(4 \\ 16 \end{array}$$

We obtain the quotient 4, as a guess at the second part of the root. Trial:

$$\begin{array}{lr} & 496 \\ \text{B} & 120 \times 4 = \underline{480} \\ & 16 \\ \text{C} & 4 \times 4 = \underline{\ 16} \\ & 0 \end{array}$$

In A, then, we have subtracted 60^2; in B, $2 \times 60 \times 4$; and in C, 4^2; consequently, we have altogether subtracted $(60^2 + 2 \times 60 \times 4 + 4^2)$, or $(60 + 4)^2$, or 64^2, and as there is no remainder, $64^2 = 4096$, $\therefore$ 64 is the required square root. In other words, if the two parts of the root are respectively a and b, we must subtract from the square $a^2 + 2 \times a \times b + b^2$ to entitle us to say that we have subtracted $(a + b)^2$ (§ 4).

Find the square root of 339889. The root lies between 100 and 1000 (§ 2). Trial shews that it also lies between 500 and 600. Call 500 the first part of the root, or a.

$$\begin{array}{rr} & 339889 \\ a^2 = 500^2 = & \underline{250000} \\ & 89889 \\ 2 \times a \times b = 2 \times 500 \times 80 = & \underline{80000} \\ & 9889 \\ b^2 = 80 \times 80 = & \underline{6400} \\ & 3489 \end{array}$$

Now the second part, or b, must be contained more than 2×500,

(= 1000) times. Dividing, then, the remainder 89889 by 1000, we obtain for quotient 89; therefore the tens' figures is 8. Performing the two subtractions, we obtain for remainder 3489, and we have now subtracted 580^2. The third part of the root, or the units' figure, must be contained in the last remainder more than 2×580 (= 1160) times. Dividing 3489 by 1160, or considering 580 as our new a, we find 3 for the units' figure.

$$
\begin{array}{rr}
 & 339889 \\
a^2 = 580^2 = & \underline{336400} \\
 & 3489 \\
2 \times a \times b = 2 \times 580 \times 3 = & \underline{3480} \\
 & 9 \\
b^2 = 3^2 = & \underline{9} \\
 & 0
\end{array}
\qquad 1160)\,3489\,(3
$$

Now we have subtracted 583^2 with remainder 0, and 583 is the required square root.

Find the square root of 12866569.

1st step. The root lies between 3000 and 4000. Subtract 3000^2.

$$
\begin{array}{rl}
12866569 & \\
\underline{9000000} & \\
6000)\,3866569 & (600 \\
\underline{3250000} & 5 \\
7000)\,616569 & (80 \\
\underline{566400} & \\
7160)\,50169 & (7 \\
\underline{50169} & \\
0 &
\end{array}
$$

2nd step. Divide 3866569 by 2×3000, and it appears that the hundreds' figure is 6. Subtract $2 \times 3000 \times 600 + 600 \times 600 = 3960000$. As this, however, is more than the remainder, it cannot be subtracted, whence we see that 6 is too large a figure for the hundreds' place. Take 5, and subtract $2 \times 3000 \times 500 + 500 \times 500 = 3250000$.

3rd step. Divide 616569 by $2 \times (3000 + 500) = 2 \times 3500$. This gives 8 as the tens' figure. Subtract $2 \times 3500 \times 80 + 80 \times 80 = 566400$.

4th step. Divide 50169 by $2 \times (3000 + 500 + 80) = 2 \times 3580 = 7160$. This gives 7 as the units' figure. Subtract $2 \times 3580 \times 7 + 7 \times 7 = 50169$, and we have no remainder, and the square root required is 3587.

EXHIBITION OF THE SUCCESSIVE STEPS.

1st step ...			*$\sqrt{12866569}$	$\overset{a}{3000}+\overset{b}{500} \parallel \overset{a}{3500}+\overset{b}{80} \parallel \overset{a}{3580}+\overset{b}{7}$
	$a^2=$	3000^2	9000000	
2nd step ...	$2\times a\times b=$	3000000	3866569	
	$b^2=$	250000	3250000	
3rd step ...	$2\times a\times b=$	560000	616569	
	$b^2=$	6400	566400	
4th step ...	$2\times a\times b=$	50120	50169	
	$b^2=$	49	50169	*Ans.* 3587.
			0	

This process may be contracted by the following considerations and methods:

a. In the 2nd step we had to take 500 first 2×3000 times, and then 500 times; in all, we had to subtract $500\times(2\times 3000+500)$ which could have been done at once by subtracting it 6500 times. In the 3rd step, we had to subtract 80 first 2×3500 times, and then 80 times, or altogether 7080 times. In the 4th step, 7 had to be subtracted $2\times 3580+7$ times, or altogether 7167 times.

b. The multiplication and subtraction can be done together as in division.

c. By substituting dots for ciphers, it will become evident that the figures will receive their true values by mere juxtaposition.

First contraction.

6...	$\sqrt{12866569}$ =	3...
5..	9......	5..
65.. × 5..	3866569	8.
70..	325....	7
8.	616569	3587
708. × 8.	5664..	
716.	50169	
7	50169	
7167 × 7	.....	

Second contraction.

	$\sqrt{12'86'65'69}$ = 3587
65	3 86
708	6165
7167	50169
	.....

* The symbol $\sqrt{}$ represents *r*, the initial letter of the word *radix*, root.

RULE:

a. Mark off the given number into "periods" of two digits, beginning with the units' figure (§ 2).

b. Find the nearest square root [3] below the 1st period [12], and the figure thus found is the first digit of the answer.

c. Subtract the square of this number [9] from the first period [leaving remainder 3], and bring down the next period [86].

d. Double the first figure of the root and use this [6] as the trial divisor.

e. Divide all the figures but the last in the second line [38] by this trial divisor; place the quotient [5] to the right both of the part of the root already found [35] and of the divisor [65]; multiply this by the new figure [5] and subtract [remainder 61].

f. Bring down the next period [65]; double the part of the root already found [35], and again use the result [70] as trial divisor.

g. Divide all the figures but the last in the third line [616] by this trial divisor, and write the quotient [8] to the right both of the root [358] and of the trial divisor [708]; multiply this by the new figure found [8], and subtract, &c.

Find the square root of 1144992021849.

	$\sqrt{1'14'49'92'02'18'49} = 1070043$
20*7	14 49
214004	92 02 18
2140083	6 42 02 49

EXERCISE LXI.

Find square roots of:

(1) 4.	(8) 289.	(15) 285970396644.
(2) 49.	(9) 3249.	(16) 501264.
(3) 100.	(10) 15129.	(17) 1607448649.
(4) 900.	(11) 582169.	(18) 41605800625.
(5) 160000.	(12) 61009.	(19) 9610000.
(6) 25000000.	(13) 956484.	(20) 123454321.
(7) 169.	(14) 68492176.	(21) 2892816758847744.

* When the trial divisor gives a 0 for quotient, care must be taken before bringing down the next period to place a cipher both in the root and in the trial divisor.

(22) A general arranges 8649 men in a solid square; how many men are there in each line?

(23) What is the length of the side of a square field 10 acres in extent?

§ 7. The square root of a fraction is found by finding the square root of the numerator and that of the denominator. For $\frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$, $\therefore$ $\sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}$.

§ 8. Find the square root of 4317.

$$\begin{array}{rl} & \sqrt{43'17} = 65 \\ 125 & 717 \\ & 92 \text{ over.} \end{array}$$

From this it appears that 65^2 is less while 66^2 is greater than 4317. The square root then lies between 65 and 66.

Exercise LXII.

(1) A miser wished to arrange 5000 sovereigns in a square. What would be the length of each side, and how many sovereigns would be over?

(2) How many more sovereigns would he want to have one more sovereign in each side of the square?

(3) What must be subtracted from each of the following numbers to leave for remainder the greatest square each contains: 8000, 80000, 3492, 75912, 25601.

(4) Find the next exact square below 56135, 82060, 10000000, 123456789, 777777, 4853741.

§ 9. Find $\sqrt{7}$.

It lies between 2 and 3. Error, if either be taken, < 1.

Try $2\frac{1}{2}$; $\frac{5}{2} \times \frac{5}{2} = \frac{25}{4} = 6\frac{1}{4}$; too little.

Try $2\frac{3}{4}$; $\frac{11}{4} \times \frac{11}{4} = \frac{121}{16} = 7\frac{9}{16}$; too much.

$\therefore$ $\sqrt{7}$ lies between $2\frac{1}{2}$ and $2\frac{3}{4}$. Error, if either be taken, $< \frac{1}{4}$.

Try $2\frac{5}{8}$; $\frac{21}{8} \times \frac{21}{8} = \frac{441}{64} = 6\frac{57}{64}$; too little.

Try $2\frac{11}{16}$; $\frac{43}{16} \times \frac{43}{16} = \frac{1849}{256} = 7\frac{57}{256}$; too much.

$\therefore$ $\sqrt{7}$ lies between $2\frac{5}{8}$ and $2\frac{11}{16}$. Error, if either be taken, $< \frac{1}{16}$; and so on to within any assigned degree of accuracy.

§ 10. Instead of approximating by binary fractions, it is more convenient to work with decimals.

$\sqrt{7}$ lies between 2 and 3. Error, if either be taken, < 1.

Call $\sqrt{7} = \sqrt{\frac{700}{100}}$* $= \frac{\sqrt{700}}{10}$ (§ 7).

$$\begin{array}{ll} & \sqrt{7'00'} = 26 \\ 46 & \;\;3\,00 \\ & \;\;\;\;24 \end{array}$$

$\therefore \frac{\sqrt{700}}{10}$ lies between 2·6 and 2·7. Error, if either be taken, < ·1.

Call $\sqrt{7} = \frac{\sqrt{70000}}{100}$.

$$\begin{array}{ll} & \sqrt{7'00'00'} = 264 \\ 46 & \;\;3\,00 \\ 524 & \;\;\;\;24\,00 \\ & \;\;\;\;\;\;3\,04 \end{array}$$

$\therefore \frac{\sqrt{70000}}{100}$ lies between 2·64 and 2·65. Error, „ < ·01; and so on to any assigned degree of accuracy.

Mod. op.:

Find $\sqrt{7}$ to within $\frac{1}{1000000}$, i.e. to 6 places.

Call $\sqrt{7} = \frac{\sqrt{7000000000000}}{1000000}$.

$$\begin{array}{ll} & \sqrt{7'00'00'00'00'00'00'} = 2645751 \\ 46 & 3\,00 \\ 524 & \;\;2400 \\ 5285 & \;\;\;\;30400 \\ 52907 & \;\;\;\;\;\;397500 \\ 529145 & \;\;\;\;\;\;\;\;2715100 \\ 5291501 & \;\;\;\;\;\;\;\;\;\;6937500 \\ & \;\;\;\;\;\;\;\;\;\;1645999 \end{array}$$

Ans. 2·645751.

The twelve ciphers might obviously be omitted from the first line and yet brought down two at a time, care being taken to put the decimal point in the root before bringing down the first pair of ciphers.

Find $\sqrt{11\frac{25}{64}}$.

$\sqrt{11\frac{25}{64}} = \sqrt{\frac{729}{64}} = \frac{\sqrt{729}}{\sqrt{64}} = \frac{27}{8} = 3·375.$

$$\begin{array}{ll} & \sqrt{7'29'} = 27 \\ 47 & 3\,29 \\ & \;\;\cdots \end{array}$$

* *Even* powers of ten are chosen as denominators because they are exact squares (§ 2).

or $\sqrt{11\frac{25}{64}} = \sqrt{11\cdot390625}$.

$$\sqrt{11'\cdot39'06'25} = 3\cdot375$$

63	2 39
667	5006
6745	33725
	.....

Ans. 3·375.

When the first pair of figures after the decimal point [39] was brought down, we were finding the root of $\frac{1139}{100}$, and hence our answer was $\frac{33}{10}$, i.e. 3·3; the next pair yielded hundredths, and so on. In marking off the periods, then, we must, as before, *begin from the units' place*, i.e. the decimal point, and mark both ways.

Find $\sqrt{57309\frac{15}{41}}$ to 5 places.

$\sqrt{57309\frac{15}{41}} = \sqrt{57309\cdot3658\dot{5}}$.

$$\sqrt{5'73'09'\cdot36'58'53'} = 239\cdot39374...$$

43	1 73
469	4409
4783	18836
47869	448758
478783	1793753
4787867	35740465
47878744	222539685
	31024709
	&c.

This root cannot be conveniently found by vulgar fractions, the denominator 41 not being a square.

§ 11. The successive divisors continually become longer, in consequence of the addition of a figure to the right, which figure, however, becomes of less and less importance. Instead, then, of adding two figures to the remainder and one to the divisor, we may simply cut off one figure at a time from the divisor (p. 134), and proceed by simple division. We shall then get one place in the root for every figure in the divisor cut off. Accordingly, we may begin to cut off as soon as more than half the number of significant figures required have been obtained.

Find $\sqrt{\frac{1}{26}}$ to 12 places.

$\frac{1}{26} = \cdot 0\dot{3}8461\dot{5}$.

$$\sqrt{\cdot 03'84'61'53'\ldots} = \cdot 196116135138\cancel{1}$$

29	284
386	2361
3921	4553
39221	63284
392226	2406361
3922321	5300553
$39\cancel{22322}$	1378232
	201535
	5419
	1497
	320
	6

Ans. ·196116135138.

Exercise LXIII.

I. Find the square root to 4 places of:

(1) 3. (2) 19. (3) 11. (4) $5\frac{1}{8}$. (5) $7\frac{3}{10}$.

II. Find to 8 places the square root of:

(1) 2, ·2, ·02, ·002.
(2) 16, 1·6, ·16, ·016, ·0000016.
(3) $\cdot 1\dot{6}$, $\cdot \dot{1}\dot{6}$, $\cdot \dot{0}1\dot{6}$.
(4) $\cdot \dot{4}$, $\cdot \dot{1}9753086\dot{4}$, $\cdot 02\dot{7}$.

III. Find by vulgar fractions and by decimals to 4 places the square root of:

(1) $3\frac{1}{16}$. (2) $1\frac{64}{225}$. (3) $51\frac{21}{25}$. (4) $15\frac{10}{81}$.

IV. Find to 12 places the square root of:

(1) $\frac{5}{8}$. (2) $\frac{7}{19}$. (3) $\frac{9}{1700}$. (4) $\frac{1}{2}$. (5) $\frac{1}{3}$. (6) $\frac{1}{30}$. (7) $\frac{1}{300}$. (8) $\frac{1}{400}$. (9) $\frac{1}{4000}$. (10) $\frac{5}{317}$.

V. Simplify to 3 places:

(1) $\sqrt{59} + \sqrt{\frac{7}{8}}$.
(2) $\sqrt{\cdot \dot{2}} - \sqrt{\cdot 2}$.
(3) $\sqrt{144^2 + 17^2}$.
(4) $\sqrt{113^2 - 112^2}$.
(5) $\sqrt{1 + \sqrt{3}}$
(6) $\sqrt{2 - \sqrt{2}}$
(7) $\frac{1}{\sqrt{2}}$.*
(8) $\frac{2}{\sqrt{11}}$.
(9) $\frac{1}{\sqrt{\cdot 001}}$.
(10) $\sqrt{5 + \sqrt{5 + \sqrt{5}}}$

* $\frac{1}{\sqrt{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \sqrt{\frac{1}{2}} = \sqrt{\cdot 5}$, which find.

VI. Find x from the following (3 places):

(1) $45 : x = x : 80$.

(2) $1 : x = x : 2$.

(3) $x : 20 = 245 : x$.

(4) $1\frac{18}{35}$, x, $8\frac{4}{7}$, three numbers in G. P. (p. 140).

EXERCISE LXIV.

The following require a slight knowledge of Geometry. Results to be found to 4 places:

(1) A rectangular room is 24 ft. long and 18 ft. broad. Find the diagonal of the floor.

(2) A tower is 180 ft. high; I stand 19 ft. away from the base. How far am I in a straight line from the top of the tower?

(3) The disc of a pendulum 85 inches long touches in its sweep two points, A and B, which are in the same horizontal line, and are 1 inch above the lowest position of the disc. How far are they apart?

(4) In the 3rd question, suppose the length of the pendulum to be 41 inches, and A and B to be 18 inches apart, how far are they above the lowest position of the disc?

(5) How long must a ladder be to reach to the top of a house 60 ft. high, when its foot is placed 11 ft. from the wall?

(6) Given in a right-angled triangle:

a. Hypotenuse 200, base 70; find perpendicular.

b. Hypotenuse 1, perpendicular $\frac{1}{\sqrt{2}}$; find base.

c. Base $\sqrt{3}$, perpendicular 1; find hypotenuse.

(7) The foot of a column 200 ft. high is 300 ft. from the base of a house 75 ft. high. Find the distance of the top of the column from the top of the house.

(8) Find the diagonal of a square whose side is 100 feet.

(9) Find the side of a square whose diagonal is 100 feet.

(10) Find the length of the perpendicular from the vertex to the base of an equilateral triangle whose side is 100.

(11) Find the side of an equilateral triangle, if the perpendicular from the vertex is 100.

(12) Find the side of a square equal to a rectangle whose sides are 588 and 507 feet.

§ 12. CUBE ROOT.

If the square of a given number be multiplied by the given number, the product is called the CUBE of the given number, and the given number is called the CUBE ROOT of the product.

$5^2 = 25$; $25 \times 5 = 125$; 5^3 or 125 is the cube of 5; and 5 is the cube root of 125, which is written $\sqrt[3]{125}$.

Roots	1,	2,	3,	4,	5,	6,	7,	8,	9.
Cubes	1,	8,	27,	64,	125,	216,	343,	512,	729.

§ 13.

$$1^3 = 1$$
$$10^3 = 1000$$
$$100^3 = 1000000$$
$$1000^3 = 1000000000,$$

&c., &c.

from which we see that a power of 10 is cubed by trebling its ciphers; and reasoning analogous to that in § 2 shews that:

Numbers with	1 figure	have in their cubes	1, 2 or 3	figures,
„	2 figures	„	4, 5 or 6	„
„	3 „	„	7, 8 or 9	„
	⋮		⋮	

Hence we shall mark off the given cube into periods of *three* figures, beginning with the units' figure.

§ 14. The expression, the cube of a number, is derived from the fact, that to find the volume of a *cube* we multiply the number of units in one side by itself, and the product again by the same number. (Part II. Ch. IV. § 6.)

P 2

§ 15. We have seen that if $A = a + b$, $A^2 = a^2 + 2 \times a \times b + b^2$ (§§ 3, 4). Similarly, it is shewn in books on Algebra, that $A^3 = a^3 + 3 \times a^2 \times b + 3 \times a \times b^2 + b^3$, or $a^3 + (3 \times a^2 + 3 \times a \times b + b^2) \times b$, thus:

$$\begin{aligned}
37^3 &= 37 \times 37 \times 37 = 37^2 \times 37 \\
&= (30^2 + 2 \times 30 \times 7 + 7^2) \times (30 + 7) \\
&= (30^2 + 2 \times 30 \times 7 + 7^2) \times 30 + (30^2 + 2 \times 30 \times 7 + 7^2) \times 7 \\
&= 30^3 + 2 \times 30^2 \times 7 + 7^2 \times 30 + 30^2 \times 7 + 2 \times 30 \times 7^2 + 7^3 \\
&= 30^3 + 3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3 \\
50653 &= 27000 + 18900 + 4410 + 343.
\end{aligned}$$

Hence to subtract 37^3 we shall have to subtract $30^3 + 3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3 = 30^3 + (3 \times 30^2 + 3 \times 30 \times 7 + 7^2) \times 7$, and after subtracting 30^3, we have from the remainder to subtract $7 \times (3 \times 30^2 + 3 \times 30 \times 7 + 7^2)$. Of the quantity in brackets, 3×30^2 is the largest term, and may therefore be used as trial divisor.

Find $\sqrt[3]{50653}$.

$$\begin{array}{rrl}
 & & \sqrt[3]{50653}\,(30 + 7 \\
3 \times 30^2 = & 2700 & 27000 \\
3 \times 30 \times 7 = & 630 & \overline{23653} \\
7^2 = & \underline{49} & \underline{23653} \\
 & 3379 \times 7 & \cdots\cdots
\end{array}$$

Here 2700, the trial divisor, is contained 7 times in the remainder 23653.

$$\begin{array}{rrl}
 & & \textbf{Contraction.} \\
3 \times 30^2 = & 27.. & \sqrt[3]{50'653} = 37 \\
3 \times 30 \times 7 = & 63. & 23\,653 \\
7^2 = & \underline{49} & \cdots\cdots \\
 & 3379 &
\end{array}$$

Here the trial divisor 27 is contained at least* 7 times in all but the last two figures of the remainder (236).

EXERCISE LXV.

Find the cube root of:

(1) 68921.	(4) 592704.	(7) 3375.
(2) 110592.	(5) 389017.	(8) 10648.
(3) 205379.	(6) 300763.	(9) 54872.

* A first inspection would lead us to try 8, but the addition of $3 \times 30 \times 8$ and 8^2 would have made the divisor too great.

Find the cube root of 42028039032·832.

		$\sqrt[3]{42'028'039'032\cdot832}$	$=3000+400+70+6+\cdot8$
3000^3	=	27.........	
		15 028	First remainder.
First trial divisor:			
3×3000^2	=27......		
$3\times3000\times400$	= 36.....		
400^2	= 16....		
	3076....×4..	12 304......	
		2 724 039	Second remainder.
Second trial divisor:			
3×3400^2	= 3468....		
$3\times3400\times70$	= 714...		
70^2	= 49..		
	353989..×7.	2 477 923...	
		246 116 032	Third remainder.
Third trial divisor:			
3×3470^2	= 361227..		
$3\times3470\times6$	= 6246.		
6^2	= 36		
	36185196×6	217 111 176	
		29 004 856·832	Fourth remainder.
Fourth trial divisor:			
3×3476^2	= 36247728		
$3\times3476\times\cdot8$	= 8342·4		
$\cdot8^2$	= ·64		
	36256071·04×·8	29 004 856·832	
		....	*Ans.* 3476·8.

Remarks:

(1) The second figure of the root, 400, is found by dividing the 1st remainder, 15028039032·832, by the 1st trial divisor, 27000000, or, which is the same, by dividing 150.. by 27; but then the new figure is called 4 and not 400; similarly, the 3rd figure, 7, is found by dividing 27240.. by 3468; the 4th figure, 6, by dividing 290048.. by 36247728, and so on.

(2) The 1st trial divisor, 27, is 3 times the square of the 1st figure; the 2nd, 3468, is 3 times the square of the part already found, 34. Now:

$$\begin{aligned}
34^2 &= 30^2 + (2\times 4\times 30) + 4^2, \text{ and}\\
3\times 34^2 &= (3\times 30^2) + (3\times 2\times 4\times 30) + (3\times 4^2)\\
&= (3\times 30^2) + (2\times 3\times 4\times 30) + (3\times 4^2)\\
&= (3\times 30^2 + 3\times 4\times 30 + 4^2) + (3\times 4\times 30) + 4^2 + 4^2\\
&= 3076 + 360 + 16 + 16.
\end{aligned}$$

∴ Hence this 3468 can be formed by adding the three preceding lines together and adding in another 4^2. Similarly, the third trial divisor 361227 is 3×347^2. Now:

$$\begin{aligned}
347^2 &= 340^2 + (2\times 340\times 7) + 7^2, \text{ and}\\
3\times 347^2 &= (3\times 340^2) + (3\times 2\times 340\times 7) + (3\times 7^2)\\
&= (3\times 340^2) + (2\times 3\times 340\times 7) + (3\times 7^2)\\
&= (3\times 340^2 + 3\times 340\times 7 + 7^2) + (3\times 340\times 7) + 7^2 + 7^2\\
&= 353989 + 7140 + 49 + 49,
\end{aligned}$$

and can therefore be obtained by adding another 49 to the three lines last obtained.

(3) Having found the new figure of the root by the help of a trial divisor, the line below that trial divisor is found by multiplying the old part of the root by 3 times the new figure. This can be done in one line unless the new figure be 8 or 9, in which case the old part must be first multiplied by 3 aside. (Part I. p. 98.)

Contracted form.

$$\begin{array}{r}
27\\ \hline
\left.\begin{array}{r}36\\ 16\\ \hline 3076\\ 16\end{array}\right\}\\ \hline
3468\\ \hline
\left.\begin{array}{r}714\\ 49\\ \hline 353989\\ 49\end{array}\right\}\\ \hline
361227\\ \hline
\left.\begin{array}{r}6246\\ 36\\ \hline 36185196\\ 36\end{array}\right\}\\ \hline
36247728\\
83424\\
64\\ \hline
3625607104
\end{array}
\qquad\qquad
\begin{array}{l}
\sqrt[3]{42'028'039'032\cdot 832} = 3476\cdot 8\\
\;\;15028\\
\;\;\;\;2724039\\
\;\;\;\;\;\;246116032\\
\;\;\;\;\;\;\;\;29004856832\\
\;\;\;\;\;\;\;\;\cdots\cdots\cdots\cdots
\end{array}$$

Find the cube root of 127268840262·941343.

$\sqrt[3]{127'268'840'262\cdot941'343} = 5030\cdot07$

7500	2 268 840
450	5 313 262 941 343
9	....
754509	
9	
7590270000	
1056300	
49	
759037563049	

1st trial divisor, 75; 1st remainder, 2268; 75 in 22.., 0', which we place after 5 in the root. The first part of the root is now 50, and $3 \times 50^2 = 7500$. Hence a 0 in the root requires 00 in the trial divisor.

§ 16. To find the cube root of a fraction, reduce the fraction to a decimal to 3 times the required number of places, unless the denominator happen to be obviously an exact cube, in which case find cube roots of numerator and denominator separately.

Find cube root of $\frac{1}{50}$ to 4 places.

$\sqrt[3]{\cdot020} = \cdot2714\ldots\ldots$

12	12000
42	317000
49	97489000
1669	9190056......
49	
2187	
81	
1	
219511	
1	
220323	
3242	
16	
22064736	

Ans. ·2714.

EXERCISE LXVI.

I. Find the cube root of:

(1) 884736. (5) 115145914625.
(2) 40353607. (6) ·017173512.
(3) 1191016. (7) ·000000004096.
(4) 8108486729. (8) ·000064481201.

II. Find the cube root (to 4 places) of:

(1) $\frac{343}{512}$. (3) $\frac{12}{27}$. (5) $\frac{5}{6}$. (7) $\frac{6}{7}$. (9) ·01.
(2) $\frac{1}{729}$. (4) $\frac{41}{32}$. (6) $15\frac{2}{3}$. (8) ·1. (10) ·001.

III. (1) If 12167 dice are piled up into a solid cube, how many dice will there be in each edge?

(2) Find the area of the surface of a cube whose volume is 3 cubic yds., 10 ft., 216 in.

§ 17. PROPERTIES OF SQUARES AND CUBES.

Some integers have square roots which are themselves integers, such as 4, 9, 16...1024, &c. (§ 1). The question arises whether the numbers between any two of these squares have a fractional square root, or no square root at all. Take any number between 4 and 9, say 6; its square root, if it have any, lies between 2 and 3, and must consequently be an improper fraction, say $\frac{a}{b}$. Now $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$; since a and b are prime to one another, we have in squaring the terms introduced no common factor, and $\therefore \frac{a^2}{b^2}$ is also at lowest terms and consequently fractional, and $\therefore$ not 6. Hence the square of a fraction cannot be integral, or, which is the same thing, an integer cannot have a fractional square root.

The number of exact squares is infinite; but there are within any assigned limits many more numbers *not* having exact square roots than there are of exact squares.

These remarks *(mutatis mutandis)* apply also to exact cubes.

§ 18. The square of a number ending in 1 or 9 ends in 1, $\because 1 \times 1 = 1$ and $9 \times 9 = 81$.

The square of a number	ending in	2 or 8	ends in	4.
„	„	3 or 7	„	9.
„	„	4 or 6	„	6.
„	„	5	„	5.
„	„	0	„	00.

Hence no exact square can end in 2, 3, 7, 8, or an odd number of ciphers.

A cube may end in any digit, but if it end in 0, it must end in a number of ciphers divisible by 3.

The cube of a number ending in		1	ends in	1,
"	"	2	"	8,
"	"	3	"	7,
"	"	4	"	4,
"	"	5	"	5,
"	"	6	"	6,
"	"	7	"	3,
"	"	8	"	2,
"	"	9	"	9.

§ 19. Terminating decimals can only be squares if the number of places be even, and then they follow the rule of § 18.

Terminating decimals can only be cubes if the number of places be divisible by 3.

§ 20. Recurring decimals may or may not be exact squares or cubes; thus:

$$\sqrt{\cdot\dot{4}} = \sqrt{\tfrac{4}{9}} = \tfrac{2}{3}, \text{ exact.}$$

$$\sqrt{\cdot\dot{5}} = \sqrt{\tfrac{5}{9}} = \frac{\sqrt{5}}{3}, \text{ not exact.}$$

$$\sqrt[3]{\cdot\dot{2}9\dot{6}} = \sqrt[3]{\tfrac{296}{999}} = \sqrt[3]{\tfrac{8}{27}} = \tfrac{2}{3}, \text{ exact.}$$

$$\sqrt[3]{\cdot\dot{2}9\dot{7}} = \sqrt[3]{\tfrac{297}{999}} = \sqrt[3]{\tfrac{23}{111}}, \text{ not exact.}$$

§ 21.

If $a > 1$, $a^2 > a$, and $a^3 > a^2$; $\therefore$ if $b > 1$, $\sqrt{b} < b$, $\sqrt[3]{b} < \sqrt{b}$.

If $a < 1$, $a^2 < a$, and $a^3 < a^2$; $\therefore$ if $b < 1$, $\sqrt{b} > b$, $\sqrt[3]{b} > \sqrt{b}$.

§ 22. If an exact square end in 6, the tens' figure is odd; if in any other figure, the tens' figure is even.

§ 23. If any number is not an exact square we can find two numbers differing from one another by as small a quantity as we please, of whose squares one shall be greater and the other less than the given number. This we have called finding the square root to so many places of decimals. For example:

$\sqrt{2}$ lies between	1·0	and 2·0	to the nearest integer,
„	1·4	and 1·5	to one place,
„	1·41	and 1·42	to two places,
„	1·414	and 1·415	to three „
„	1·4142	and 1·4143	to four „

and so on.

§ 24. Here and in § 9 we have in our expressions tacitly assumed that there are such things as $\sqrt{2}$, $\sqrt{7}$, &c., although they are not expressible by exact numbers. Such language is justified by the following considerations, which will be understood by those who have been able to work Exercise LXIV.

If the side of a square be 1, the square on the diagonal is 2 (Euc. I. 47), and the length of the diagonal is $\sqrt{2}$. $\sqrt{2}$, then, is an actual quantity, and measurements of the line continually increasing in accuracy by stages of the decimal scale will yield the above approximations.

Similarly, lines can be found whose length is $\sqrt{3}$, $\sqrt{7}$, &c.

§ 25. $3^4 = 3 \times 3 \times 3 \times 3 = 3^2 \times 3^2 = (3^2)^2$. Hence the square root of the square root is the 4th root, or $\sqrt[4]{a} = \sqrt{\sqrt{a}}$.

$$3^6 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^2 \times 3^2 \times 3^2 = (3^2)^3.$$

Again, $3^6 = 3^3 \times 3^3 = (3^3)^2$, $\therefore \sqrt[6]{a} = \sqrt[3]{\sqrt{a}} = \sqrt{\sqrt[3]{a}}$. Hence the cube root of the square root, or the square root of the cube root, is the 6th root. Similarly, the cube root of the cube root is the 9th root; the cube root of the 4th root, or the 4th root of the cube root, is the 12th root; and so on.

CHAPTER XII.

VARIOUS.

§ 1. Number of decimal places to be retained in finding a continued product.

The factors of a continued product may be proper or improper fractions, and our *modus operandi*, in finding an approximate pro-

duct with a given limit of error, must depend on their nature in this respect.

$2\frac{7}{8} \times 4719\frac{14}{15} \times 6\frac{4}{11} \times \cdot008\dot{3}$.

By vulgar fractions : $\frac{23}{8} \times \frac{70799}{15} \times \frac{70}{11} \times \frac{1}{120} = \frac{11398639}{15840}$.

70799	1584·0) 1139863·9 (719·611
161	3106
70799	15223
424794	9679
70799	1750
11398639	1660
	76

Ans. $719\frac{9679}{15840}$, or to three places 719·611.

By decimals to three places : $2\cdot875 \times 4719\cdot9\dot{3} \times 6\cdot\dot{3}\dot{6} \times \cdot008\dot{3}$.

It would seem that the factors might be taken in any order ; let us take them from right to left.

6·4	4719·9333
38 00	3 50
51	236997
2	14160
·053 first product.	250·157 second product.
	5782
	500314
	200126
	17511
	1251
	719·202 third product.

This result is greatly inaccurate even in the first place of decimals. The first product ·053 is accurate to the third place, but is nevertheless inaccurate for want of the succeeding places. Now this inaccuracy is multiplied by nearly 4720 in finding the second product, and this again is nearly trebled in finding the last product. These accumulated inaccuracies, only slightly compensated for by occasional over-estimates, have given as final result the very appreciable error of ·409.

These errors can be avoided thus : Arrange the factors in descending order, and move the several points so as to make each factor but the first a proper fraction. We may then, to ensure accuracy, keep

one more than the assigned number of places all through (see N.B. p. 122), and each successive multiplication by a proper fraction will reduce the error.

Thus the problem $4719{\cdot}9\dot{3} \times 6{\cdot}\dot{3}\dot{6} \times 2{\cdot}875 \times {\cdot}008\dot{3}$,

becomes $47199{\cdot}\dot{3} \times {\cdot}\dot{6}\dot{3} \times 2{\cdot}875 \times {\cdot}008\dot{3}$

$= 471993{\cdot}\dot{3} \times {\cdot}\dot{6}\dot{3} \times {\cdot}2875 \times {\cdot}008\dot{3}$,

or better still, $4719{\cdot}9\dot{3} \times {\cdot}8\dot{3} \times {\cdot}\dot{6}\dot{3} \times {\cdot}2875$.

$$
\begin{array}{r}
4719{\cdot}9333 \\
33333\ 338 \\
\hline
37759466 \\
1415980 \\
141598 \\
14160 \\
1416 \\
142 \\
14 \\
1 \\
\hline
3933{\cdot}2777 \\
36363\ 636 \\
\hline
23599666 \\
1179983 \\
235997 \\
11800 \\
2360 \\
118 \\
24 \\
1 \\
\hline
2502{\cdot}9949 \\
5\ 782 \\
\hline
5005990 \\
2002395 \\
175209 \\
12515 \\
\hline
719{\cdot}61\cancel{0}\cancel{9} \\
1
\end{array}
$$

Ans. 719·611.

§ 2. Compound Interest.*

When the interest is not drawn, but is added from time to time, as it becomes due, to the principal, and is calculated for each inter-

* Compound interest can only be worked satisfactorily by logarithms.

val on the amount thus increased, the total charge made for the loan is said to be reckoned at COMPOUND INTEREST.

Find the compound interest on £256 at 4 % for $1\frac{1}{2}$ years, reckoned half-yearly.

In this case $\frac{r \times t}{100}$ will for each half-year be $\frac{4 \times \cdot 5}{100} = \cdot 02$; hence each successive amount will have to be multiplied by ·02.

1st method.

Mod. op.:

Original principal	£256
Interest for 1st period, £256 × ·02..............	5·12
Amount due at end of 1st period	261·12
Interest for 2nd period, £261·12 × ·02	5·2224
Amount due at end of 2nd period	266·3424
Interest for 3rd period, £261·3424 × ·02	5·3268...
Amount due at end of 3rd period	271·6692
Deduct original principal..........................	256
Compound interest	£15·6692

Ans. £15. 13*s*. $4\frac{3}{4}$*d*.

Multiplying any number by ·02 and adding the product to the original number, is equivalent to multiplying at once by 1·02, which can be done in one line (Part I. p. 99), thus :

256
261·12
266·3424
271·6692...
256
15·6692 *Ans.* £15. 13*s*. $4\frac{3}{4}$*d*.

2nd method. Find the amount at compound interest of £845. 12*s*. 8*d*. at 5 % for 5 years, reckoned yearly.

First find compound interest on £1 at 5 % for 4 years. Uniform multiplier $1 + \frac{r \times t}{100} = 1 \cdot 05$.

Original principal........	£1·
Amount at end of 1st year	1·05
,, 2nd ,, ...	1·1025
,, 3rd ,, ...	1·157625
,, 4th ,, ...	1·21550625
,, 5th ,, ...	1·2762815625

Now multiply this fifth line by the principal, $845 \cdot 6\dot{3}$:

$$\begin{array}{r} 1 \cdot 276281 \times 845 \cdot 6\dot{3} \\ \underline{33\ 36548} \\ 10\ 21025 \\ 51051 \\ 6381 \\ 766 \\ 38 \\ \underline{4} \\ 1079 \cdot 265 \end{array}$$

Ans. £1079. 5*s.* $3\frac{3}{4}$*d.*

A computer who has often to calculate compound interest would find it useful to make tables similar to the above, carried out further according to need, for various rates and periods.

Exercise LXVII.

(1) Find the compound interest on :

a. £585 for 2 years, reckoned quarterly, at 4 % per annum.

b. £1000 for 10 years at 5 %, reckoned yearly.

c. £60 at 5 % for 1 year, reckoned monthly.*

d. £145. 17*s.* 6*d.* for $2\frac{1}{2}$ years at $3\frac{1}{2}$ %, reckoned half-yearly.

e. £624. 12*s.* 8*d.* for 5 years at $3\frac{7}{8}$ %, reckoned yearly.

f. £815. 13*s.* 9*d.* at £2. 17*s.* 9*d.* %, for 3 years, reckoned half-yearly.

(2) Find the difference between simple and compound interest reckoned half-yearly, on £850 for 3 years at $5\frac{1}{2}$ % per annum.

(3) Find the difference between simple and compound interest reckoned yearly, on £738. 15*s.* at $4\frac{3}{4}$ % per annum, for 4 years.

(4) Find the difference in compound interest between reckoning yearly, half-yearly and quarterly, on £1000 for 3 years at 4 % per annum.

§ 3. Converse of Compound Interest.

What sum of money will in 3 years amount to £3000 at $3\frac{1}{2}$ % per annum compound interest, reckoned half-yearly ?

Uniform multiplier $1 + \frac{r \times t}{100} = 1 \cdot 0175$.

* This question can by the second method be worked thus :

(Amount of £1 for 1 month $)^2$ = amount for 2 months
(Amount ,, 2 months$)^2$ = ,, 4 ,,
(Amount ,, 4 months$)^3$ = ,, 12 ,, (Cf. Ch. XI. § 25.)

Table for £1 (keeping 8 places).

Original principal			....	£1·
Amount at end of 1st period			....	1·0175
,,	2nd	,,	$(1{\cdot}0175)^2$......	1·03530625
,,	3rd	,,	$(1{\cdot}0175)^3$......	1·05342410...
,,	4th	,,	$(1{\cdot}0175)^4$......	1·07185903...
,,	5th	,,	$(1{\cdot}0175)^5$......	1·09061656...
,,	6th	,,	$(1{\cdot}0175)^6$......	1·10970235...

If £1 amounts to £1·10970235,
x ,, £3000.

1·10970235 : 3000 = 1 : x.

```
110970235) 30000000 (2703·427
            7805953
             38037
              4746
               307
                85
```

Ans. £2703. 8*s.* $6\frac{1}{4}$*d.*

Exercise LXVIII.

(1) What sum of money will amount to:

a. £500 at 5 % per annum compound interest, reckoned yearly, in 4 years?

b. £750 at 4 %, reckoned six-monthly, in $2\frac{1}{2}$ years?

c. 1000 guineas at $3\frac{1}{2}$ % in 5 years, reckoned yearly?

§ 4. Equation of Payments.

Occasionally it is proposed to find the time at which several debts due at different times might be discharged by a single payment of their sum total.

Find the "equated time" for the following bills:

£800 due in 4 months at 5 %.
£560 ,, 6 ,,
£375 ,, 9 ,,

$$\text{Present value of the } £800 = \frac{800 \times 100}{100 + 5 \times \cdot\dot{3}} = 786{\cdot}8852$$

$$\text{,, ,, } £560 = \frac{560 \times 100}{100 + 5 \times \cdot 5} = 546{\cdot}3414$$

$$\text{,, ,, } £375 = \frac{375 \times 100}{100 + 5 \times \cdot 75} = 361{\cdot}4458$$

$$\text{,, ,, } £1735 = \qquad 1694{\cdot}6724$$

In what time, then, will £1694·6724 amount to £1735 at 5 %? (Formula V. p. 184.)

$$I = 1735 - 1694{\cdot}6724 = 40{\cdot}3276$$

$$t = \frac{100 \times 40{\cdot}3276}{1694{\cdot}6724 \times 5} = {\cdot}476 \text{ of a year} = 173 \text{ days.}$$

Ans. 173 days.

This is reckoned rigorously, by discount; but in commerce, when such questions occur, a mere average is struck:

$$\frac{800 \times 4 + 560 \times 6 + 375 \times 9}{1735} = 5{\cdot}72 \text{ months (about 171 days).}$$

§ 5. Simplification of Complex Decimals.

Simplify $\dfrac{4{\cdot}375}{4{\cdot}5} \times \dfrac{41}{5{\cdot}125} \times \dfrac{7{\cdot}875}{{\cdot}0625}$.

The product of the numerators will have 6 places, that of the denominators 8. If we expel all the points, treating the several numbers as integers, we shall have to multiply the result by 100. This gives:

$$\frac{\overset{\substack{7\\ \cancel{35}\\ \cancel{175}\\ \cancel{875}}}{\cancel{4375}} \times \cancel{41} \times \overset{\substack{7\\ \cancel{63}\\ \cancel{315}\\ \cancel{1575}}}{\cancel{7875}} \times 100}{\underset{\cancel{9}}{\cancel{45}} \times \underset{\substack{\cancel{1025}\\ \cancel{205}\\ \cancel{41}}}{\cancel{5125}} \times \underset{\substack{\cancel{125}\\ \cancel{25}\\ 5}}{\cancel{625}}} = \frac{4900}{5} = 980.$$

Simplify $\dfrac{4{\cdot}75}{5{\cdot}625}$. $\quad \dfrac{4{\cdot}75}{5{\cdot}625} = \dfrac{4{\cdot}75 \times 8}{5{\cdot}625 \times 8} = \dfrac{38{\cdot}\cancel{00}}{45{\cdot}\cancel{000}}$.

* A terminal 5 in a decimal will disappear by multiplication by 2.

„	25, 75	„	„	4.
„	125, 375, 625, 875	„	„	8.

(Cf. Part I. Ch. IX. § 19.)

This, applied to the former simplification, gives:

$$\frac{4{\cdot}375 \times 8}{4{\cdot}5 \times 8} \times \frac{41 \times 8}{5{\cdot}125 \times 8} \times \frac{7{\cdot}875 \times 8}{{\cdot}0625 \times 8} = \frac{35 \times \cancel{41} \times 8 \times 63}{36 \times \cancel{41} \times {\cdot}5} = \frac{35 \times \overset{2}{\cancel{8}} \times \overset{7}{\cancel{63}} \times 2}{\underset{\cancel{4}}{\cancel{36}} \times {\cdot}\cancel{5} \times \cancel{2}} = 980.$$

* See foot-note, p. 175.

Reduce to a vulgar fraction at lowest terms ·267578125.

$$\frac{267578125 \times 8}{1000000000 \times 8} = \frac{2140625\cancel{000} \times 8}{8000000\cancel{000} \times 8} = \frac{17125\cancel{000} \times 8}{64000\cancel{000} \times 8} = \frac{137\cancel{000}}{512\cancel{000}}. \quad \text{(Cf. p. 102.)}$$

Ans. $\frac{137}{512}$.

A competent knowledge of the results of Chap. VI. will enable the computer to make shrewd guesses in the simplification of fractions containing *recurring* decimals.

§ 6. G. C. M. AND L. C. M. OF DECIMALS.

Find G. C. M. and L. C. M. of 5·625, 2·88, 3·6, 2·8125.

Equalize the decimal places: 5·6250, 2·8800, 3·6000, 2·8125.

Find G. C. M. and L. C. M. of these numbers, disregarding decimal points, and mark off the 4 places from each result. (Cf. Part II. Ch. IV. § 4.) *Ans.* G. C. M. ·0225; L. C. M. 360.

§ 7. INTERCONVERSION OF FRACTIONS IN DIFFERENT SCALES.

Reduce $\cdot\dot{1}2\dot{3}$ (quinals) to duodecimals (to 6 places).

$$s = \cdot 123123\ldots\ldots$$
$$5^3 \times s = 123\cdot 123123\ldots\ldots$$
$$(5^3 - 1) \times s = 444 \times s = 123 \qquad s = \tfrac{123}{444} \text{ in the quinary scale,}$$
$$= \tfrac{38}{124} \Big| \tfrac{19}{62} \text{ in the decimal scale.}$$

$$\begin{array}{lll}
\text{Dec.} & \text{Dec.} & \text{Duodec.} \\
62) & 19 \times 12 & (\cdot 38166e7 \\
 & 228 & \\
 & \ \ 42 \times 12 & \\
 & 504 & \\
 & \ \ \ 8 \times 12 & \\
 & 96 & \\
 & \ \ 34 \times 12 & \\
 & 408 & \\
 & \ \ 36 \times 12 & \\
 & 432 & \\
 & \ \ 60 \times 12 & \\
 & 720 & \\
 & \ \ 38 \times 12 & \\
 & 456, \text{ \&c.} &
\end{array}$$

Ans. ·38167$\cancel{0}$.

§ 8. DUODECIMALS.

The Duodecimal scale, both integral and fractional, can be usefully applied to the calculation of small areas and volumes. The

foot is taken as the unit; the inch, $\frac{1}{12}$ of 1 foot, is in the *first* place, and the "part" or 12th of an inch, or $\frac{1}{12^2}$ of 1 foot, in the *second*. Hence in square measure, the square foot being the unit, the square inch is in the second place, and the square part in the fourth; and in cubic measure, the cubic foot being the unit, the cubic inch is in the third place, and the cubic part in the sixth place.

Given length, 1 ft., 11 in., 3 pts.; breadth, 1 ft., 7 in., 6 pts.; find area. 1·*e*3 × 1·76. (Part I. Ch. X.)

$$
\begin{array}{r}
1\ e\ 3 \\
\underline{1\ 7\ 6} \\
e\ 7\ 6 \\
1\ 1\ 6\ 9 \\
\underline{1\ e\ 3} \\
3{\cdot}1\ 9\ 4\ 6
\end{array}
$$

$3{\cdot}1946 = 3$ sq. ft., $(1\times12+9)$ sq. in., $(4\times12+6)$ sq. pts.
$= 3$ sq. ft., 21 in., 54 pts.; $\frac{54}{144}\big|\frac{3}{8}$,
$= 3$ sq. ft., $21\frac{3}{8}$ sq. in.

or by contracted multiplication, keeping 2 places, to be correct to square inches.

$$
\begin{array}{r}
1{\cdot}e\ 3 \\
\underline{6\ 7\ 1} \\
1\ e\ 3 \\
1\ 1\ 7 \\
\underline{{\cdot}1\ 0} \\
3{\cdot}1\ t
\end{array}
$$

$3{\cdot}1t = 3$ sq. ft. 22 in. Error, $\frac{5}{8}$ of 1 sq. in.

§ 9. International Calculations.*

a. Length.

	Multiply by:	or Divide by:
To turn yards into metres	·9143862	1·09363
,, ,, Prussian ells	1·371	·7294
,, ,, Austrian ells	1·1743	·8516
,, ,, Spanish varas	1·0784	·9273
,, ,, Portuguese varas	·8318	1·2022
,, ,, Russian arsheens	1·2857	·7
,, miles into kilometres	1·609315	·6213824
,, ,, Prussian miles	·21364	4·6807
,, ,, Austrian miles	·21212	4·7142

* These calculations are mainly based on Woolhouse's Measures, Weights and Moneys of all Nations.

	Multiply by:	or Divide by:
.rn miles into Spanish leagues	·23723	4·2152
„ Portuguese miles......	·7821	1·2786
„ Russian versts	1·50852	·6629

Surface.

ırn sq. yds. into centiares (sq. metres)	·83612	1·196
, acres into hectares	·404671	2·471143
, „ Prussian morgen	1·5848	·631
, „ Austrian joch	·70308	1·4223
, „ Portuguese geiras......	·69187	1·4453

Capacity.

ırn gallons into litres.................	4·54345	·220097
, „ Prussian eimer	·06614	15·118
, „ Austrian eimer......	·08027	12·4572
, „ Spanish cantaros ...	·28264	3·538
, bushels into hectolitres	·363476	2·751211
„ „ Austrian metzen ...	·15762	6·3442
„ „ American bushels...	1·03152	·96944
„ quarters into Russian chetverts .	1·3863	·7213

. Weight.

urn lbs. av. into lbs. troy	1·21527	$\cdot 82\dot{2}8571\dot{4}$
„ grains into grammes	·06479895	15·4323487
„ lbs. av. into kilogrammes.........	·45359265	2·2046212
„ „ Prussian pounds ...	·96983	1·0311
„ „ Austrian pounds ...	·80959	1·2352
„ „ Spanish pounds ...	·9858	1·0144
„ „ Portuguese pounds .	·98828	1·01186
„ „ Russian pounds ...	1·10786	·90264
„ cwts. into quintals (metric) ...	·508023765	1·9684118
„ „ Prussian zentner ...	·987428	1·012732
„ „ Austrian zentner ...	·90674	1·102857
„ „ Portuguese quintals	·864745	1·1564107
„ „ Spanish quintals ...	1·10414	·90571
„ „ Russian berkowitz .	3·102	·32237
„ tons into French tons (milliers)	1·01604753	·9842059
„ „ Russian packen	2·068008	·483557

§ 10. Use of the Tables.

Express 9 tons, 13 cwt., 1 qr., 16 lbs., as metric tons or milliers.

9 tons, 13 cwt., 1 qr. (£9. 13*s.* 3*d.*) = 9·6625 tons.
16 lbs. ($\frac{1}{7}$ of ·05) = ·007142...
9·669642 ,,

Multiply by 1·015652:

```
 1·015652
 46 9669
 --------
  91409
   6094
    609
     91
      6
 --------
 9·8209
      1
```

Ans. 9·821 milliers. Error < 1 kilog.

or divide by ·9846875:

```
·9846875) 9·669642 (9·820
           807454
            19704
               10
```

Ans. 9·820.

For conversion of foreign weights and measures to English, *multiply* by the given divisors or *divide* by the given multipliers, i. e. reverse the processes.

Express 419·785 kilometres as miles.

Multiply by ·6213824:

```
 419·785
 4283 126
 --------
 2518710
   83957
    4198
    1259
     335
       8
       2
 --------
 260·8469 × 8
   6·775  × 20
  15·5··  × 11
 170·5
```

Ans. 260 m., 15 fur., 170½ yds.

or divide by 1·609315:

```
1·609315) 419·7850 (260·8470
           97 9220        8
            1 3631    6·776
               757       20
               113   15·52·
                 1       11
                      170·7
```

Ans. 260 m., 15 fur., 170$\frac{7}{10}$ yds.

Error < 7 inches.

§ 11. Coinage.

The ratios for interconversion of the coinage of different nations can only be given at par, and would accordingly be nearly useless, as that rate rarely prevails. If the exchanges* are given, Chain Rule will apply.

* By "exchange" is meant the rate at which bills due in foreign countries are negotiable here, which fluctuates with the state of trade, &c.

d the value of £647. 11*s.* 1*d.* in reis (Portugal), at 57½*d.* per

x	647·55416̇
·239583̇	1

2·3958333) 6475·5417 (2702·835
1683 8750
6 7917
2 0000
834
115

Ans. 2702·835 reis = 2702 milreis, 835 reis.

ıd the arbitrated exchange* between London and Lisbon, if rei = 5·95 fr., and £1 = 25·15 fr.

£x	5·95 fr.
fr. 25·15	£1

25·15) 5·950 (·236
920
165
14

Ans. 4*s.* 8¾*d.*, or 56¾*d.* per milrei.

12. Find the price per yard in English money at 7·85 fr. per e; also at 4·27 fr., at 5·19 fr., and at 6·45 fr. per metre; exge, 25·75.

› turn metres into yards, we multiply by 1·09363. Multiplying, , this ratio by the exchange, we obtain the common divisor for ;everal prices given.

25·75
363 901
25750
2318
77
15
1
28·161

28·161) 7·850 (·2787
2 218 9
247
22 *Ans.* 5*s.* 7*d.*

28·161) 4·270 (·1516
1 454
46
18 *Ans.* 3*s.* 0½*d.*

28·161) 5·190 (·1843
2 374
121
9 *Ans.* 3*s.* 8¼*d.*

28·161) 6·450 (·229
818
225
2 *Ans.* 4*s.* 7*d.*

The "arbitrated exchange" is the rate realized by remitting to a place abroad, directly, but *viâ* some other place or places.

Thus common multipliers or divisors can, as occasion requires, be established for the working of classes of questions.

§ 13. For ordinary purposes francs are valued at 25 per £, which leads to easy calculations.

The following are ratios compounded on the Metric System, the money being calculated *at par* (fr. 25·2215 = £1), i.e., according to the intrinsic value of the coins.

To find in one operation the price in English money :

					Multiply by : or	Divide by :
a.	Per yard,	given price per	metre in	francs	·03625	27·5862
b.	„ sq. yard,	„	sq. metre	„	·03315	30·1649
c.	„ acre,	„	hectare	„	·016044	62·3275
d.	„ gallon,	„	litre	„	·1765735	5·66335
e.	„ bushel,	„	hectolitre	„	·0144113	69·3897
f.	„ lb. av.,	„	kilo	„	·01798432	55·6038536
g.	„ cwt.,	„	quintal	„	·020142488	49·646298
h.	„ ton,	„	millier	„	·040282976	24·823149

Find the cost in English money of 1 yd. at fr. 7·72 per metre.

$$\begin{array}{r} \cdot03625 \\ \underline{277} \\ 2538 \\ 254 \\ \underline{7} \\ \cdot2799 \end{array}$$

Ans. 5*s.* 7¼*d.*

Exercise LXIX.

(1) Express 73 lbs., 12 oz. av. in :

a. Kilogs. *c.* Austrian pounds.

b. Russian pounds. *d.* Prussian pounds.

(2) Express 25·625 French tons as :

a. English tons. *b.* Russian packen.

(3) Find the cost in English money per yd. at fr. 2·45 per metre.

(4) „ „ per cwt. at fr. 16·75 per quintal.

(5) „ francs per French ton at 14*s.* 9*d.* per Engl. ton.

(6) How many litres in $4\frac{1}{2}$ Austrian eimer?

(7)* The average height of the barometer at Paris is 76 centimetres. Reduce this to inches correctly to three places of decimals.

* Nos. 7 to 20 are copied from an excellent tract on the Metric System, by J. J. Walker, Esq., M.A.

Reduce the hectare to a. r. sq. po. yds. exactly. How much t differ from $2\frac{1}{2}$ acres?

Supposing the quadrant of the meridian of Paris to be 6213 fur., 23 po., 4 yds. in length, calculate the length of the metre hes to five places.

) The French post-office allows 7·5 grammes for a single post- the English $\frac{1}{4}$ of an oz. av. By how many grains does the h exceed the English allowance?

.) How many hectolitres = 1 cubic metre? A tank is $37\frac{1}{2}$ deci- s long, by 25 wide and 18 deep. How many gallons would ld?

2) If wine be sold at 457 francs the cask of 7 hectolitres, what d be the corresponding price in *s. d.* of the bottle of 6 to the n?

3) How many bushels = 1 hectolitre? If wine be sold at ancs the hectolitre, what would be the corresponding price per el in *s. d.*?

4) The length of the tunnel through Mont Cenis will be about 2 kilometres. What will this be in miles?

5) The diameter of bore and weight of a piece of French ord- e are given as 27 centimetres and 22,000 kilogrammes. Give corresponding measure and weight in inches and cwt.

6) A building plot in Paris is offered for sale at 75 francs per netre. What would be the corresponding price per sq. yard in . *d.*?

.7) The distance between two stations on a Belgian railway is down as $7\frac{1}{2}$ kilometres, and is done by a train in 12 minutes. at is the rate per hour in miles?

18) The pressure of the atmosphere at the average height of the ometer is $14\frac{3}{4}$ lbs. av. to the sq. inch. What would be the cor- onding pressure in kilogrammes to the square centimetre?

19) If the sack of flour of 157 kilogrammes be sold at 53·75 fr., at would be the corresponding price of the quartern (4 lbs.) in *s.d.*?

20) The rent of a farm of 23·25 hectares is 1225 francs. What he rate per acre in £. *s. d.*?

CHAPTER XIII.

ARITHMETICAL COMPLEMENTS.*

§ 1. The ARITHMETICAL COMPLEMENT of a number is the quantity by which it falls short of the next power of 10 above it, or, in other words, the DEFECT of the number from that power of 10. Thus the A. C. or defect of 3792 is $10000 - 3792 = 6208$. (Cf. Part I. Ch. I. § 11.)

§ 2. This defect is obtained by complementing to 9 each digit but the units' figure, which must be complemented to 10. By its means several arithmetical processes can be performed in shapes very different from those in common use, and these are useful when the A. C. is small. The same principle can be extended to cases where complementing only the right-hand portion of a number will yield a number not greater than 12 followed by ciphers.

§ 3. Addition can be performed by the aid of subtraction. Thus "to add 9, add ten and *deduct* one." (Part I. p. 11, line 10.) Similarly, to add 3995, it is shortest to add 4000 and *subtract* 5. Conversely, subtraction can be performed by the aid of addition; thus to subtract 9, subtract 10 and *add* 1; and to subtract 3995, subtract 4000 and *add* 5.

§ 4. Multiplication, which is a series of additions (Part I. p. 44), can accordingly be performed by a series of subtractions, or by a multiplication and subtraction. Thus to multiply by 998, multiply by 2 and subtract the product from the multiplicand three places to the right. (Part I. p. 108.) Similarly, to multiply by 3995, we may multiply by 4 and subtract 5 times the multiplicand three places lower down.

7243·849516 × 39·95.

$$\begin{array}{l} \underline{7243849516} \times 4\ldots \\ 289753\ 98064\ldots \\ 289391{\cdot}7881642\mathrm{0} \end{array} \qquad \textit{Ans. } 289391{\cdot}7881642.$$

* The matter of this chapter is mostly drawn from a pamphlet by George Suffield, Esq., M.A., Clare College, Cambridge.

$\cdot\dot{7}\dot{6} \times 79\cdot993$.

$$\begin{array}{r} \underline{2345\dot{7}\dot{6}} \times 8 \ldots . \\ 08 \\ 187661\dot{4}|1414 \\ 1876449\;9382 \\ 76 \end{array}$$

Ans. $187644\cdot9937\dot{6}$.

ing: 48′, carry 4; 56, 60′, carry 6; hence (Ch. VI. § 16) $08+6=\dot{1}\dot{4}'$, ; 40, 46′, carry 4; 32, 36′, carry 3, &c. Put on 1414 for the (. . . .) and 7 times the multiplicand, thus: 42 and 2′, 44, carry 4; 49, 53, and 8′, y 6; hence (Ch. VI. § 16) $82-6=\dot{7}\dot{6}'$, carry 6; 35, 41, and 3′, 44, carry 32, and 9′, 41, carry 4, &c.

Division, which is a series of subtractions (Part I. p. 76), performed by a series of additions.

ide 123456789 by 7384. $7384 = 8000 - 616$; hence for every deducted we have deducted 616 too much, which must there- ep by step be added to get the true remainder.

$$\begin{array}{lrl} & 8000)\,123456789\,(166 \left.\begin{matrix}\\1\end{matrix}\right\} \left.\begin{matrix}18\\1\end{matrix}\right\} & \\ & 4\ldots & \\ 1\times616\ldots\ldots & \underline{616} & \\ & 49616 & \text{1st remainder.} \\ & 1\ldots & \\ 6\times616\ldots\ldots & \underline{3696} & \\ & 53127 & \text{2nd remainder.} \\ & 5\ldots & \\ 6\times616\ldots\ldots & \underline{3696} & \\ & 8823 & \text{3rd remainder } (a) \\ & 0\ldots & \\ 1\times616\ldots\ldots & \underline{616} & \\ & 14398 & \text{3rd remainder } (b) \\ & 6\ldots & \\ 1\times616\ldots\ldots & \underline{616} & \\ & 70149 & \text{4th remainder.} \\ & 6\ldots & \\ 8\times616\ldots\ldots & \underline{4928} & \\ & 11077 & \text{5th remainder } (a) \\ & 3\ldots & \\ 1\times616\ldots\ldots & \underline{616} & \\ & 3693 & \text{5th remainder } (b) \end{array}$$

Ans. 16719 times and 3693 over.

REMARKS :

(1) The successive remainders are the same as by the common method.

```
7384)123456789(16179
      49616
       53127
        14398
         70149
          3693
```

(2) 8000 in 12345, once and 4345 over, ∴ 7384 in 12345, once and $4345 + 1 \times 616 = 4961$ over; 8000 in 49616, 6 times and 1616 over, ∴ 7384 in 49616, 6 times and $1616 + 6 \times 616 = 5312$ over; and so on.

(3) This method gives for third remainder 8823, which contains 8000, and *a fortiori* 7384, one more time, altering the 6 in the quotient to 7, and the remainder, after an addition of 1×616, to 1439. A similar case occurs in finding the last remainder. But for this "doubtful case," which is peculiar and inherent to it, this method would be preferable to ordinary long division.

Divide 123456789 by 499.

```
        5..)123456789(247408
             2..
2×1......    2
             2365
              3..
4×1......     4
              3696             499)123456789(247408
               1..                   2365
7×1......      7                      3696
               2037                    2037
                0..                     4189
4×1......       4                        197
                4189
                 1..
8×1......        8
                 197 over.
```

This mode of division is called SYNTHETIC DIVISION, from *synthesis*, which means putting together. The ordinary mode might, per con-

:alled "Analytic Division." The 5 is called the SYNTHETIC
:.

ination of this synthetic division shews that to find the
:cessive remainders, each figure of the quotient is added to
.dend two places to the right of that figure from which it
ained. All this can be done mentally thus:

5..)123456789
247308 and 197 over.
4

ing: 5 in 12, 2′ (beneath 4), carry 2; in 23, 4′ (beneath 5), carry 3; 2) 36, 7′, carry 1; in (15+4) 19, 3′, carry 4; in (46+7) 53, 10′ (alter- previous 3 into 4′), carry 3; in (37+4) 41, 8′, and (189+08) 197 over.

Ans. 247408 and 197 over.

may find decimal places:

5..)123456789
247308·3947895791583, &c.
4

ling (continued): In 41, 8′, carry 1; in (18+0) 18, 3′, carry 3; in (39+ 9′, carry 2; in (20+3) 23, 4′, carry 3; in 39, 7′, carry 4; in 44, 8′, &c.

When the synthetic divisor is 1, the doubtful case occurs so that the following form will be found more convenient.

/ide 27689954372 by 999.

/iding by 1000 we obtain for quotient 27689954 and 372 hence dividing by 999 we obtain the same quotient and 27689954 over, which being again divided by 999 will yield) and 954 + 27689 over; and so on.

Mod. op.:

27689954	372
27689	954
27	689
	27
27717672	042
	2
	·044

Ans. $27717672\cdot\dot{0}4\dot{4}$.

ıe 2 carried from the sum of the remainders to the quotients ; 2000 yields another complementary 2 to the remainder, h is accordingly 44. And $\frac{44}{999} = \cdot\dot{0}4\dot{4}$.

§ 7. A slight extension of this method will adapt it to division by synthetic divisor 1, when A. C. is any number by which we can multiply mentally in one line. (Part I. Ch. IX. §§ 3, 4, 5, 6.)

Divide 27689954372 by 992.

Here the compensation is each time 8 times the quotient.

```
27689954|372
  221519|632
    1772|152
      14|176
        |112
--------|---
27913260|444
        |  8
         ---
         452   Ans. 27913260 and 452 over.
```

To decimalize $\frac{452}{992}$.

```
1...) 4|520
        32
      -----
       5|520
          40
       ------
        5|600
           40
        ------
         6|40, &c.
```

which can be worked nearly mentally:

```
452
  52
   60
    40
     48
     512
      160
       608
        128
         288
          896
          9024*
            320
             224      Ans. ·45564516129032...
```

* Here carriage makes the 8 a 9, and consequently we have to *add in* one more 8 to the 24, making the remainder 32.

48146 ÷ 9983. (A. C. 17.)

```
    17364|8146
       29|5188
         | 493
    -----|----
    17394|3827
           17
          ----
 1....)3844
         91
        5046
         5450
           85
           918
            265  Ans. 17394·38505492.
```

tiplication by 17 is done in one operation by the method of . Ch. IX. § 3, and the same *mod. op.* can be used in all cases ch §§ 4, 5, 6, in the same chapter can be applied to the A. C.

568994 ÷ 9799. (A. C. 201.)

```
    32756|8994
      658|3956
       13|2258
         |2613
    -----|----
    33428|7821
         | 201
          ----
 1....)8022
        1828
          481
          6418
           5386
            4865
             9454, &c.      Ans. 33428·8186549...
```

3. All numbers not divisible by 2 or 5 can be multiplied so as eld a product of the form 9999...... (Fermat's Theorem, p. 148). ke 37. To obtain 9 in the units' place we multiply by 7.

$$\begin{array}{r} 7 \times 37 = 259 \\ 20 \times 37 = \underline{740} \\ 999 \end{array}$$

change the tens' figure 5 into 9 we have to add 4 tens, and therefore multiply by 2 tens; hence the required multiplier is If, then, we have to divide by 37, we multiply the dividend 7 and divide synthetically by (1...).

Take 67. Multiplier 597 obtained from the right, figure by figure.

```
  597
   67
  ---
  469
 649
399
```

Hence $67 \times 597 = 39999$, synthetic divisor 4...., without requiring to carry out the whole process.

§ 9. Decimalize $\frac{3258692}{177}$. $177 = 3 \times 59$.

```
3 ) 3258692            by common division.
6.) 1086230·666...     by synthetic divsion.
      18410 689265536, &c.
```

Decimalize $\frac{14}{183}$.

Mod. op.: Multiplier, 53

```
  183
  ---
  549
 969
```

$14 \times 53 : 9699$. (A.C. 301, S.D. 1....)

```
1....)·07420
       6307
        4876
         9964
        502349
          2650
           7102
            3127
```

Ans. ·07650273.

Decimalize $\frac{5}{19}$. (S. D. 2.)

```
2.) 5·
     ·263157894 | 7, &c.
```

The remaining figures can be obtained by complementing (Ch. VI. § 10), and it is most convenient to write the second half of the period under the first:

```
2.) 5·
     ·263157894 |
      736842105
```

We know that we have arrived at the complementing stage because the remainder 14 complements the dividend 5 with respect to the divisor 19. (Ch. VI. § 12.)

10. Synthetic Division Subtractive.

7 ÷ 71.

7.)827
1164788732394...

rding: 7 in 8, 1′, carry 1; in (12 − 1) 11, 1′, carry 4; in (47 − 1) 46, 6′, 4; in (40 − 6) 34, 4′, carry 6; in (60 − 4) 56, 8, 7′, carry 7; in (70 − 7) ′, carry 7, &c.

11. Synthetic division can of course be applied to other scales of tion by complementing to the next higher power of the radix.

564 ÷ 266 (septenary).

3..)13564
34325541604, &c.

rding: 3 in ten, 3′, carry 1; in (7 + 5) 12, 4′; in (6 + 3) 9, 3′; in 8, 2′, 2; in (2 × 7 + 3) 17, 5′, carry 2, &c.

Exercise LXX.

Simplify:

) $\{ 3 \times (49993 + 2 \times 3997) \} + 9998$.

) $\{ 9 \times (49993 - 2 \times 3997) \} \times 9998 \times 701$.

) $793\cdot\dot{7}1\dot{8} \times 3\cdot997$.

) $\cdot\dot{1}5384\dot{6} \times 3\cdot9$.

) $\cdot\dot{3}8461\dot{5} \times 91000$.

) By synthetic division (integral):

a. 7473684 ÷ 19, 29, 39, 49, 59.

b. 226543817 ÷ 199, 4999, 399, 3990.

c. 8543764333 ÷ 99, 999, 9999.

d. 4623814 ÷ 98, 997, 9996.

e. 54376146 ÷ 983, 9799, 9898.

) Decimalize by synthetic division, completing the period: $\frac{319}{41}$, , $\frac{5284}{17}$, $\frac{5284}{19}$, $\frac{5284}{23}$, $\frac{5284}{31}$, $\frac{5284}{47}$, $\frac{5284}{495}$, $\frac{5284}{2997}$.

) To 12 places: 73 ÷ 9995; 47 ÷ 989; 119 ÷ 9992.

12. These methods will save much time to those who have ired aptitude in detecting opportunities.

CHAPTER XIV.

MISCELLANEOUS EXAMPLES.

(Answers to money sums to be brought to the nearest farthing; other problems to three places, unless otherwise specified.)

(1) By vulgar fractions and by decimals, work the following, and shew that the results coincide:

a. $8\frac{7}{20} + 3\frac{11}{25} + 9\frac{4}{5} + 10\frac{3}{8} + 4\frac{15}{32}$.

b. $6\frac{2}{5} - 3\frac{15}{32}$.

c. $\frac{3}{8} \times \frac{1}{10} \times 2\frac{1}{2} \times {\cdot}001$; $\frac{5}{16} \times \frac{1}{8} \times \frac{1}{25} \times 1000000$.

d. $14 \div 3\frac{1}{5}$; $2\frac{1}{2} \div \frac{8}{25}$; $\frac{1}{4} \div \frac{2}{625}$.

(2) Find the limits of the following:

$\cdot\dot{1}3\dot{5}$, $\cdot\dot{1}5\dot{3}$, $\cdot0\dot{1}3\dot{5}$, $\cdot2\dot{1}3\dot{5}$, $\cdot\dot{0}13\dot{5}$, $\cdot013\dot{5}$, $\cdot015\dot{3}$.

(3) By decimal calculations only, find the following:

a. The cost of $5437\frac{5}{12}$ articles at £1. 13*s.* $10\frac{1}{4}$*d.* each.

b. „ 287694 articles at $9\frac{5}{16}$*d.* each.

c. „ 157 tons, 13 cwts., 2 qrs., 13 lbs., at £38. 10*s.* 8*d.* per ton.

d. The dividend on £347. 18*s.* 10*d.*, at 13*s.* $9\frac{7}{8}$*d.* in the £.

e. The profit on £468. 17*s.* 5*d.*, at £9. 13*s.* $4\frac{1}{4}$*d.* per cent.

f. The brokerage on £1267. 10*s.*, at $2\frac{1}{2}$ per mille.

g. The premium on £768, at $3\frac{1}{8}$ per cent., insured so as to recover both goods and premium in case of loss.

(4) Extract the square root of 191810·713444.

(5) Find to 4 places the difference between the square and cube roots of 32·14.

(6) Find to 3 places $2 \times \sqrt{3} - \frac{1}{2} \times \sqrt{12} + \sqrt{27}$.

(7) Prove that $\sqrt{18\cdot\dot{7}} = 4\frac{1}{3}$.

(8) Find the discount, the simple interest, and the compound interest, on £465 for 18 months at $4\frac{1}{2}$ % (compound interest calculated half-yearly).

(9) Find a fourth proportional to $1\frac{1}{2}$, ·09, $\frac{9}{20}$.

(10) Shew that $\frac{1}{2600}$ of £10. 16s. 8d. is equal to ·002 of £2. 1s. 8d., l that ·001 of £2 $= \frac{1}{2\cdot08\dot{3}}$ of 1d.

(11) Reduce to a decimal: $\cfrac{3}{4+\cfrac{2}{3+\cfrac{2}{3}}}$

(12) Extract the square root of 272·316004.

(13) Extract to 9 places the square root of ·034.

(14) Sum the series $1 + \frac{3}{2} + 2 + \frac{5}{2} + \ldots$ to 30 terms.

(15) Sum the series $\frac{387}{1000} + \frac{387}{1000000} + \frac{387}{1000000000}$ to infinity.

(16) A can dig a certain ditch in 3 days, B in 4 days, and C in lays. How long will it take the three together to dig the ditch, l what fraction of it is dug by each?

(17) Find the amount at compound interest at $12\frac{1}{2}$ % on £819. 4s. 6 years, reckoned yearly.

(18) Extract to 5 places the cube root of $\cdot0\dot{3}\dot{4}$.

(19) If the carriage of 1 cwt., 12 lbs. for 105 miles comes to $10\frac{1}{2}d.$, what should be charged for the carriage of 8 cwt., 1 qr., lbs. for 245 miles?

(20) Simplify $\frac{5\frac{2}{3}}{7\frac{1}{2}}$ of $\frac{21\cdot25}{\cdot046875}$.

21) Find in what time £452. 10s. will amount to £644. 16s. 3d. $4\frac{1}{4}$ % per annum simple interest.

(22) Express £4. 6s. $4\frac{3}{4}d. + \frac{1}{5}$ of 1 farthing as a decimal of £5.

(23) A grocer mixes 3 cwt., 15 lbs. of sugar at $5\frac{1}{2}d.$ per lb. with cwt., 10 lbs. at 4d. per lb. At what price per lb. should he sell mixture to gain 25 %?

(24) Find G. C. M. and L. C. M. of 6·3375, 73·125, 39, 12·1875.

(25) Find the cost of 6 reams, 13 quires, 10 sheets, at £1 per m.

26) A man who has $\cdot\dot{3}$ of the profits of a partnership sells ·25 his share, and the buyer makes £89. 6s. 8d. per annum. What he yearly income of the whole business, and if the buyer pay 000 for his share, what interest does he get for his money?

(27) Sum the series:

a. $2 + 5 + 8 + 11 +$ to 20 terms.

b. $3 + \frac{3}{10} + \frac{3}{10^2} +$ to 10 terms.

(28) One sample of tea costs 3*s.* 4*d.* per lb., and another 4*s.* per lb. At how much per lb. must the mixture of 60 lbs. of the former and 25 lbs. of the latter be sold to gain $7\frac{1}{2}\ \%$?

(29) Find the side of a square grass plot which is of the same area as a rectangular grass plot 63 ft. long and 28 ft. broad.

(30) Find the difference between the simple and compound interest reckoned yearly, on £210 for 2 years at 5 %.

(31) Find the difference to 3 places between $\sqrt[3]{3}$ and $\sqrt{2}$.

(32) A carpenter makes 2 chairs in 3 days, and 3 chairs and 1 table in 8 days. In what time would he make 6 chairs and 3 tables?

(33) A man walks 80 miles; he begins by walking 4 hours a day, at the rate of 5 miles an hour, and each day increasing the number of hours by 1, he diminishes the pace by 1 mile per hour. How many hours does he walk, and how many days?

(34) If a model of a cathedral is to be made on the scale of 6 ft. to $\frac{3}{4}$ of an inch, what will be the dimensions in the model of a tower 120 ft. high, a roof 50 ft. long, and a floor 40 square yards in area?

(35) Simplify:

a. $\left\{ \frac{3}{5} \div \left(\frac{4}{25} - \frac{1}{9}\right) \right\} + \left\{ 1 \div \left(\frac{2}{5} - \frac{1}{3}\right) \right\} - \left\{ 1 \div \left(\frac{2}{5} + \frac{1}{3}\right) \right\}$

b. $\left(\frac{2}{3} + \dfrac{\frac{4}{5} - \frac{2}{3}}{1 + \frac{8}{15}}\right) \times \frac{5}{6} \div \left(1 - \frac{2}{3} \times \dfrac{\frac{4}{5} - \frac{2}{3}}{1 + \frac{8}{15}}\right)$

c. $\dfrac{\frac{2}{3} \times \sqrt{3} \times \frac{1}{2} \times \sqrt[3]{2}}{\frac{5}{8} \times \sqrt[3]{2} \times \frac{2}{7} \times \sqrt{3}}$

(36) The items of a journey on the continent are £4. 10*s.* 6*d.*, 50·75 francs, 66 thalers. Find the whole cost of the journey in English, French and German money, when £1 = 25·22 francs = $6\frac{2}{3}$ thalers.

(37) In the morning I solved $\frac{3}{4}$ of $\frac{1}{3}$ of a certain number of problems; by the end of the afternoon I had done $\frac{2}{5}$ of $\frac{2}{3}$ of the number. Suppose the whole number of problems to be 300; how many did I solve during the afternoon?

3) Multiply and divide to 4 places $\cdot\dot{3}8095\dot{2}$ by $1\cdot\dot{3}$, and verify esults by reducing the decimals to their limits.

)) How much 3 per cent. stock must I sell out to pay a debt 550, the price of stock being $94\frac{1}{8}$, and brokerage $\frac{1}{8}$ %?

)) Find accurately the value of:

a. £$\cdot003$, £$\cdot0\dot{0}\dot{3}$, £$\cdot\dot{0}0\dot{3}$, and £$\cdot00\dot{3}$.

b. $\cdot08\dot{3}$, $\cdot0\dot{8}\dot{3}$, $\cdot\dot{0}8\dot{3}$ of a ton.

c. $\cdot41\dot{6}$, $\cdot\dot{4}1\dot{6}$ of 1 lb. troy.

d. $\cdot0099$, $\cdot00\dot{9}$ of 1 lb. av.

e. $\cdot\dot{1}0\dot{8}$, $\cdot11\dot{8}$ of 1 gallon.

1) Find the reciprocal of the difference between 31·24 and $3768\dot{1}4285\dot{7}$.

2) Which is cheaper, to buy napoleons (20 fr.) at 15*s.* $10\frac{1}{2}$*d.*, ounds sterling at 25·22 fr.?

3) A kilogramme is 2·205 lbs. avoirdupois; a French ton =) kilogs. What fraction of an English ton is a French ton?

4) Find the difference between $5\frac{1}{2}$ sq. ft. and $5\frac{1}{2}$ ft. square.

5) How much stock must I sell out of the Consols when they it $93\frac{3}{4}$ (brokerage $\frac{1}{8}$), to raise a sum of £1265?

6) Find the square roots (to 4 places of decimals) of 1979649, $\frac{3}{7}$, ·25, ·35, ·000025, ·025, ·001.

17) Find the cost of a case lined with tin, 5 ft., 10 in. long, 4 ft. id, and 1 ft., 8 in. deep, inside measurement, at 2*d.* per sq. ft. for wood, and $2\frac{1}{4}$*d.* per sq. ft. for the tin.

18) If a napoleon be worth 15*s.* $10\frac{1}{2}$*d.*, find the lowest exact num- of napoleons that must be given for an exact number of English ereigns, stating the number of each.

19) Simplify to 5 places of decimals:

$$\sqrt{1024} + \sqrt{102\cdot4} + \sqrt{10\cdot24} + \sqrt{1\cdot024} + \sqrt{\cdot001024}.$$

50) Find the cost of papering a room 16 ft. long, 11 ft. wide, 10 ft. high, with paper 30 inches broad, at $7\frac{1}{2}$*d.* a yard.

51) Multiply (by the method of duodecimals) 7 ft., 5 in., 8 parts, 9 ft., 4 in., 11 parts.

(52) Express the result of the last question as square inches.

(53) Arrange the following journeys in order of rapidity:

Name of Line.	Departure from London.	Arrival at		Distance.
Great Northern	10 a.m.	Aberdeen	3 a.m.	526 m.
Great Western	9.15 a.m.	Penzance	9.30 p.m.	328 m.
North Western	10 a.m.	Carlisle	6.10 p.m.	299 m.
Great Eastern	5 p.m.	Yarmouth	9.45 p.m.	146 m.
South Western	11.10 a.m.	Weymouth	4 p.m.	147 m.

(54) The fares for these journeys are £4, £3. 10*s.*, £2. 15*s.*, £1. 10*s.*, £1. 9*s.* 6*d.* respectively. Arrange them in the order of cheapness.

(55) What is the average charge per mile?

(56) If I invest 7000 guineas in the $3\frac{1}{2}$ per cents. at 93, what is my nett income, deducting 5 % income-tax?

(57) A room costs £8 to paper. What would a room cost half as high again, half as long again, and half as broad again, with paper costing half what it did before per yd.?

(58) If 144 men can dig a trench 40 yds. long, $1\frac{1}{2}$ ft. broad, and 48 ft. deep, in 3 days of 10 hours each, how long must another trench, 5 ft. deep and $2\frac{1}{4}$ ft. broad, be, in order that 51 men may dig it in 15 days of 9 hours each?

(59) The discount on a sum due one year hence at 5 % per annum is £15. What is the sum?

(60) How many dice a third of an inch long can be packed into a box whose dimensions inside are 2, 3 and 4 ft. respectively?

(61) How many dice a third of an inch long can be packed into a cubical box $1\frac{1}{2}$ ft. long inside?

(62) Find the square root of $\cdot\dot{3}0864197\dot{5}$.

(63) A mass of lead ore weighing 800 grains troy, was found to contain ·6 grain of silver. What is the value of silver in one ton of the ore, at the rate of 5*s.* the oz. troy?

(64) If £24. 7*s.* $10\frac{1}{2}$*d.* be paid as income-tax on an income of £650. 10*s.*, what ought to be paid at the same rate on an income of £2450. 6*s.* 8*d.*? And at what rate in the £ is the tax levied?

) In how many years will £625. 10s. amount to £813. 3s., at imple interest?

) Find the value of 159 cwt., 3 qrs., 22 lbs., at £2. 12s. 6d. wt.

) Add $\frac{1}{8}$ of a guinea, £$\frac{9}{32}$, $\frac{1}{20}$ of a crown, and $\frac{15}{16}$ of a shilling, express the whole as a decimal of £1.

8) What sum of money will at $4\frac{1}{4}$ % simple interest amount years to £1497. 4s. 1d.?

9) If the price of 3 bushels of wheat is 16s. 9d., find the price qrs., 2 bus., 1 peck.

0) Find the cost of making a road, length 9 miles, 5 fur., 44 at £25. 8s. 4d. per mile.

1) In what time will a sum of money double itself at 5 %, at simple interest, (b) at compound interest, reckoned yearly?

2) If when wheat is 60s. a quarter the sixpenny loaf weigh 4 how much should be paid for 25 lbs. of bread when wheat is a quarter?

73) What decimal of an English mile is an Indian league, of ch 30 go to a degree (60 geographical miles), 1 geographical e being 1·1508 English miles?

74) A Russian gold ducat is worth 9s. 5d. sterling; 3 roubles ke a ducat, and 100 kopeks 1 rouble. Find in English money value of 315 ducats, 2 roubles, 80 kopeks.

75) Gun-metal consists of 100 parts of copper and 11 of tin. w much of each metal will there be in a cannon weighing 3 tons, wt., 1 qr., 16 lbs.?

76) Find the income arising from investing £740 in the 3 per ts. at $92\frac{1}{2}$.

(77) Reduce to lowest terms $\frac{403}{899}$, $\frac{5371}{59736}$.

(78) Find the sum of 19 terms of the series 7, 14, 21, &c.

(79) If a degree of longitude at X—— be $\frac{2}{3}$ of the length of a gree at the equator, how many miles at the latitude of X—— ll the sun pass over in a minute and a half, given that the equar is 131470565 feet long?

(80) A ship's company take a prize of £1000, which is to be divided amongst them in proportion to their pay and to their time of service; the officers, 4 in number, have 40*s.* each a month, and the midshipmen, 12 in number, have 30*s.* each a month, and they have all served six months; the sailors, 110 in number, have each 22*s.* a month, and have served 3 months. Find each man's share.

(81) An Austrian bankrupt owes a London merchant 5784 florins, and pays $68\frac{1}{2}$ %. How much is that in the £, and how much sterling money will be remitted to the Englishman (exchange 11·45 fl. = £1)?

(82) The rent-roll of a certain estate amounts to £3580 a-year. The repairs average $7\frac{1}{2}$ % per annum. Find the value of the estate at 28 years' purchase.

(83) Which is the heavier income-tax, 3 % or 7*d.* in the £?

(84) A steamer working with a given force can travel down the river at the rate of $12\frac{1}{2}$ miles an hour. Of this speed, $\frac{2}{7}$ is due to the current. How long would the steamer take to travel 15 miles up the stream?

(85) Find the rent of 204 acres, 1 rood, 20 poles, at £2. 15*s.* 9*d.* per acre.

(86) Find the *discount* on £972, due 10 months hence, at $5\frac{1}{4}$ % per annum, and shew what rate of *interest* is charged in this case.

(87) If the rations of 3264 men for 48 days cost £4787. 4*s.*, what is the cost of the rations of 5000 men for 90 days?

(88) Two persons have invested £11. 17*s.* $2\frac{3}{4}$*d.* and £17. 16*s.* $8\frac{1}{2}$*d.*, and the return is £46. 2*s.* $0\frac{3}{4}$*d.* Find within a farthing what the share of each must be.

(89) Of £121. 13*s.* $4\frac{3}{4}$*d.* and £29. 8*s.* 10*d.*, what percentage is each of the other (5 places)?

(90) Determine without any superfluous work $\sqrt{1·0097626}$ to 8 places.

(91) Find correct to one 10,000th of a unit 16·112734 × ·20708 × 1146·339 ÷ ·00007.

(92) A ship valued at £14,500 is insured at £3. 10*s.* %, and her cargo valued at £32,000 is insured at £4. 17*s.* 6*d.* %. Find the whole cost of insurance.

93) I invested £680 in Consols at $89\frac{3}{8}$; 3 days later the funds
; to $90\frac{7}{8}$. What would have been my loss of income had I
ted these 3 days, brokerage $\frac{1}{8}$ %?

94) A speculator bought in Consols at $88\frac{3}{8}$, and sold out at $91\frac{3}{4}$,
kerage $\frac{1}{8}$ % each time; his total gains amounted to £350. Find
value of the stock when bought in.

95) I bought $3\frac{1}{2}$ per cent. stock at $95\frac{1}{2}$, and after drawing one
f-yearly dividend, I sold at $92\frac{7}{8}$; my total loss of capital amounted
£8. 15*s.* Find the amount of stock I had held.

96) The prices of the 3 per cent. Consols, and Midland Railway
ck, paying $5\frac{1}{4}$ %, were quoted at $95\frac{3}{8}$ and $108\frac{1}{2}$ respectively.
ıd the difference in income from investing £100 in each.

97) Find the average of $17\frac{1}{2}$, $25\frac{1}{4}$, $96\frac{3}{8}$, 10, 0, $42\frac{3}{4}$, 56, and ex-
ss the answer decimally.

98) The income of a parish is £6529, 10*s.* 6*d.* How much in
£ will produce a rate of £150?

99) If the time after 1 p.m. is $\frac{7}{15}$ of the time before midnight,
at o'clock is it?

100) Find cube root of $1776\frac{325}{729}$.

101) Express 1 acre, 3 roods, 26 perches, as the decimal (in full)
a square mile.

102) If 120 men make an embankment $\frac{3}{4}$ of a mile long, 30
ds wide, and 7 yds. high, in 42 days, how many men would it
e to make an embankment 1000 yds. long, 36 yds. wide, and 22
high, in 30 days?

(103) A person invests £1365 in the 3 per cents. at 91; he sells
t £1000 stock when they have risen to $93\frac{1}{2}$, and the remainder
ıen they have fallen to 85. Find his gain or loss.

(104) A and B have gained £600 between them; A has to re-
ve 10 % less than B. Find their respective shares.

(105) Distribute £1250 among A, B and C, giving to A 15 %
re, and to B 12 % less than to C.

(106) Distribute £760 among A, B and C, giving to A $17\frac{1}{2}$ %
s than B's share, which is 20 % more than C's share.

(107) A bankrupt's estate amounts to £910. 3*s.* $1\frac{1}{2}$*d.* and his debts to £1875. What can he pay in the £, and what will a creditor lose on a debt of £57 ?

(108) An estate with a rental of £8790 is sold for £351,600. In order that it may yield the purchaser $3\frac{3}{4}$ % for his money, how much % must he raise the rent?

(109) A square court-yard costs £38. 10*s.* 5*d.* to pave, at 3*s.* 9*d.* per square yard. Find the length of its side.

(110) Express 37048 (decimal) in the nonary scale; also 347102 (nonary) in the decimal scale.

(111) The Hanoverian mile is 25400 Hanoverian feet long, each foot being ·9542 English feet. Find to 4 places of decimals the fraction that an English is of a Hanoverian mile.

(112) How many times will a wheel whose diameter is $3\frac{2}{7}$ feet revolve in travelling over 5 miles? (N.B. Circumference : diameter = 3·14159 : 1.)

(113) If a package weighing 7·5 cwts. be carried 125 miles for 14*s.* 7*d.*, how much will be charged for the carriage of 3 tons, 15 cwts. for a distance of 200 miles?

(114) If 770 gallons of creosote at 1*d.* per gallon have the heating power of 8·75 tons of coal at £·641$\dot{6}$ per ton, find the yearly saving in money in a factory which burns 1000 gallons a day, omitting 52 Sundays.

(115) Extract the square root of 1194·3936 and of $\frac{14\cdot 4}{16\cdot 9}$.

(116) Simplify :

$$\frac{\frac{5}{2}\times\frac{1}{1\frac{1}{4}}+1\frac{5}{7}\text{ of }\frac{1}{\frac{4}{7}}}{8}+\frac{\frac{21}{3\frac{1}{3}}\times\frac{5}{18\frac{3}{4}}\times 37\frac{1}{2}}{\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{2}\times\frac{1}{5}}}-\left(\tfrac{1}{7}\text{ of }\frac{25}{3\frac{1}{8}}\text{ of }\frac{3\frac{1}{3}}{6\frac{2}{3}}+\frac{1}{14}\text{ of }\frac{1}{1\frac{1}{3}}\right)$$

(117) Express + 40° and − 40° Fahrenheit on the Centigrade and Réaumur scales.

(118) Find the cost of $3047\frac{5}{11}$ articles at £1. 15*s.* $9\frac{1}{2}$*d.* each.

(119) What capital will at $3\frac{1}{2}$ % in three months amount to £348. 0*s.* $4\frac{1}{2}$*d.* ?

120) A bankrupt's debts amount to £27485. 10*s.* 9*d.* and his ets are worth £9328. 6*s.* 3*d.* What is he to pay on each of the owing debts: £248. 14*s.* 5*d.*; £7642. 10*s.* 6*d.*; £19. 4*s.* 2*d.*?

121) Find the cost of 47 iron plates, each weighing 127 tons, cwt., 1 qr., 19 lbs., at £50. 6*s.* 10*d.* per ton.

122) Find the value of 21 acres, 2 roods, 15 perches, at 7. 15*s.* 6*d.* per acre.

123) If the sixpenny loaf weigh 4·35 lbs. when wheat is at 5·75 llings per bushel, what weight of bread ought to be purchased 18·13 shillings when wheat is at 18·4 shillings per bushel?

124) Three gardeners working full time can plant a field in 10 rs. How long will it take them if one of them works half time?

125) A person bought into the 3 per cents. at 98, and after eiving 3 years' interest sold out at 90. How much % on the n invested did he gain or lose?

126) A Turkey carpet measuring 12 ft., 6 in., by 11 ft., 6 in., laid down on the floor of a room measuring 14 ft. by 13 ft. termine the quantity of floorcloth necessary to complete the ering of the floor, and its price, at 4*s.* per square yard.

(127) Reduce 9*s.* $11\frac{3}{4}$*d.* to the fraction of half-a-sovereign.

(128) Simplify $\frac{1 \cdot 18}{\cdot 152} \times \frac{3 \cdot 04}{2 \cdot 95} \div \cdot 00125$.

(129) Find the cost of 457 tons, 13 cwt., 3 qrs., 19 lbs., at . 17*s.* $8\frac{1}{2}$*d.* per ton. Also of 17 lbs., 9 oz., 17 dwts. troy, at . 15*s.* $10\frac{1}{4}$*d.* per oz.

(130) Find the compound interest for £55 for 1 year, reckoned arterly, at 5 % per annum.

(131) Find, without unnecessary work, to 3 places:

a. $\cdot 4\dot{0}08\dot{6} \times 16 \cdot \dot{0}5\dot{9} \times 2618 \cdot 0\dot{8}5\dot{3} \times \cdot 00035$.

b. $\cdot \dot{4}1\dot{9} \times 9 \cdot \dot{8} \times 720 \times 43 \cdot \dot{1}5\dot{6}$.

c. $3 \cdot 1415927 \ldots \times \sqrt{7000} \times \sqrt[3]{\frac{1}{3}} \times 1870$.

(132) Find the difference between the square and cube roots of 15380329 (to 1 place).

(133) Simplify $\sqrt{25\frac{44759}{117649}} - \sqrt[3]{25\frac{44759}{117649}}$. Also find the value $\sqrt[3]{\cdot 000729} - \sqrt{\cdot 000729}$; and explain why the sign $\sqrt{}$ is placed st in one case, and $\sqrt[3]{}$ in the other.

(134) A buys 134 gallons of beer for £11. 18s.; 6 gallons are lost by leakage; he sells the rest in jugs, holding $\frac{3}{8}$ of a quart, at $2\frac{1}{4}d.$ per jug. Find his total profit, and his profit %.

(135) The receipts of a railway company are apportioned in the following manner: 49 % for working expenses, 10 % for the reserve fund, a guaranteed dividend of 5 % on one-fifth of the capital, and the remainder, £40000, for division among the holders of the rest of the stock, being a dividend of 4 % per annum. Find the capital and the receipts.

(136) Find the square root of 24·2064; of 3124·81; of $2{\cdot}42064 \times 312{\cdot}481$ and of $\dfrac{2{\cdot}42064}{312{\cdot}481}$.

(137) Find the cost of paving a hall 50 yds. long by 50 ft. broad, with marble slabs 1 ft. long and 9 in. broad, the price of the slabs being £5 per dozen.

(138) Simplify $\dfrac{{\cdot}0075 \times 2{\cdot}1}{{\cdot}0175} + \dfrac{4{\cdot}255 \times {\cdot}0064}{{\cdot}00032}$.

(139) Find the square root to 8 places of decimals of 38715, 1500, 150, 7, ·7, ·07, 49, 4·9, ·49, $\cdot\dot{4}\dot{9}$, $\cdot 3402\dot{7}$.

(140) The quadrant is divided into 90^0 and also into 100 grades. Express $37\frac{1}{3}$ deg. + $50\frac{2}{9}$ gr. both as degrees and as grades.

(141) A person having £1000 invests in the 3 per cents. at $92\frac{3}{8}$, brokerage $\frac{1}{8}$; after 3 years he sells at $94\frac{5}{8}$, and again pays $\frac{1}{8}$. What did he receive as interest, and what did he gain on the whole?

(142) Divide £26. 3s. 3d. between 3 persons so that their shares may be in the proportion of £2. 18s. 6d., £1. 19s. and £1. 9s. 3d.

(143) If the price of candles $8\frac{1}{2}$ inches long be 9d. per half-dozen, and that of candles of the same thickness and quality $10\frac{1}{4}$ inches long be 1s. $4\frac{1}{2}d.$ for 9 candles, which is the cheaper kind, and how much % is lost by buying the dearer?

(144) Find the profit or loss per dozen on a quantity of wine "laid down" in 1848 at a cost of £2. 8s. per dozen, and sold in 1870 at £5. 5s. per dozen, reckoning compound interest annually at 5 %.

(145) Express $\cdot\dot{3}$ (in the quinary scale) as a decimal fraction.

46) I sold goods at a loss of $7\frac{1}{2}$ %; had I sold them at a gain $\frac{1}{2}$ % I should have realized £3. 15s. more than I actually re- d. Find the cost of the goods.

47) Multiply by duodecimals 9 ft., 7 in., 3 pts., by 5 ft., 7 in., ts., and the product by 2 ft., 7 in. What does the product me when expressed in cubic feet and inches?

48) In 1841 the population of Great Britain was 21,476,000, that of Ireland 7,310,000; in 1851 the former had increased % and the latter had decreased 12·5 %. Find the increase % ıe population of the United Kingdom.

49) Reduce $\frac{68}{157}$ to a converging fraction; give the several con- ents and the limit of error in each.

50) A certain book costs in production 2s. $4\frac{1}{2}d.$ per copy, and etail price is 7s. 6d.; the publisher allows the bookseller 25 % he retail price, and gives 13 copies to the dozen; 3900 copies printed and sold; the author is to have half the profits. How h will he receive?

51) A tenant pays a corn rent of 20 quarters of wheat and 12 arley, Winchester measure. What is the value of his rent, at being at 60s. and barley at 54s. a quarter imperial measure, oning a Winchester bushel to be $\frac{32}{33}$ of an imperial bushel.

52) Express as a decimal $\frac{117}{5^{11} \times 2^7}$.

153) A certain Building Society accepts £11. 14s. 1d. at the nning of the year in lieu of 12 monthly instalments of £1 each. at yearly payment will at this rate discharge a monthly liability :5. 17s. 10d.?

154) I wish to borrow of a Building Society £600, to be paid off 5 years by monthly instalments, paying interest at 6 % on the le sum borrowed for the whole time. What should be my ·ly payment at £11. 14s. 1d. instead of £1 per month?

155) I hold £43. 17s. 5d. Building Society stock for 7 months, the year's balance sheet shews a dividend of £11. 13s. $8\frac{3}{4}d.$ per t. per annum. What dividend should I receive?

(156) A cistern holding 820 gallons is filled in 20 minutes by 3 pipes, the first of which conveys per minute 10 gallons more, and the second 5 gallons less, than the third. How much flows through each pipe per minute?

(157) Gold is 19·3 times, and copper is 8·62 times as heavy as water. How many times as heavy as water is standard gold, which is a mixture of 11 parts of gold and 1 of copper?

(158) The annual retreat of the equinox along the ecliptic is 50″. In what time will the equinox be carried round the whole circle of the ecliptic (360^0)?

(159) In what time will the sun move through 50·1″, when it traverses 360^0 in 365 days, 6 hours, 9 min., 9·6 sec., the motion being supposed uniform?

(160) A bankrupt's stock was sold for £520. 10*s.*, at a loss of 17 % on the cost price. Had it been sold in the course of trade, it would have realized a profit of 20 %. How much was it sold below the trade price?

(161) A foreign Government contracts for three loans in different markets: the first, a 5 % loan for 20 millions; the second, a 4 % loan for 12 millions; the third, a $3\frac{1}{2}$ % loan for 10 millions. For the first the Government received £65, for the second £50, for the third £42 for every £100 stock. How much money does Government receive for all these loans, what average rate of interest is paid on the money actually received, and on which of the three loans does the Government pay the lowest rate of interest?

(162) A besieged garrison loses 5 men on the first day, 10 on the second, 15 on the third, and so on for 30 days, when the commandant, finding that he had only $\frac{1}{3}$ of his original garrison left, surrendered. How many men were there at first?

(163) A merchant sells tea to a tradesman at a profit of 60 %, but the tradesman becomes bankrupt and only pays 13*s.* 4*d.* in the £. How much per cent. does the merchant gain or lose?

(164) Find the sum which must be invested in the 3 per cents. at 90, to amount in $23\frac{1}{2}$ years to £3317 sterling, the price of the funds remaining unchanged. If we sold out at 96, how many years sooner could the required amount be realized?

5) The sidereal year consists of 365 days, 6 hours, 9 minutes, conds, reckoned in mean solar time, or of 366 days, 6 hours, ıutes, 9·6 seconds, reckoned in sidereal time. Find the ratio idereal to a solar day, to 5 places.

;6) Two shepherds, A and B, owning a flock of sheep, agree to e it; A takes 144 and B 184 sheep, paying £70 to A. Find alue of 1 sheep.

37) Among how many persons must £158. 17*s.* 3*d.* be divided, der that half of them may have 10*s.* 7*d.* each, and the other 7*s.* 2*d.* each?

68) A and B have the same sum of money; A buys equal ınts of 3 per cent. stock at 91, and of $3\frac{1}{2}$ per cent. at $97\frac{1}{2}$; B sts his money equally in the purchase of the same stocks. A's me being 1*s.* more than B's, how much money had each?

.69) Assuming a cubic foot of water to weigh 1000 oz. av., find weight of a rainfall of one inch over an acre of ground.

.70) A person has a sum invested in the 3 per cents., which he ; and invests in the $3\frac{1}{2}$ per cents. at $87\frac{1}{2}$. If his income remains same, what was the price of the 3 per cents.?

171) A takes 6 steps while B takes 7; but 4 of A's steps are al to 5 of B's. Which is the quicker walker?

172) An army lost 18 per cent. of its strength by disease and ertion, and then 14 per cent. of the remainder in battle; the nber then remaining was 84,624. Of how many did it originally ısist?

(173) A person sells £5000 Consols at $94\frac{7}{8}$, and on their rising sells £5000 more at $95\frac{5}{8}$; on their again rising he buys back the ıole £10000 at 96. What does he lose?

(174) If gold is 19·3 times as heavy as water, and copper is 8·96 nes as heavy as water, how many times its own bulk of water will rown weigh composed of 9 oz. of gold and 15 oz. of copper?

(175) Find by inspection (table, p. 152) the number of recurring d non-recurring figures in the decimalization of each of the following fractions: $\frac{1}{8}, \frac{1}{625}, \frac{1}{16\times25}, \frac{1}{2^{10}\times5^{10}}, \frac{1}{2^5\times5^3}, \frac{1}{7}, \frac{1}{56}, \frac{1}{91}, \frac{1}{21}$, $\frac{1}{\times37}, \frac{1}{2\times5\times7\times37}, \frac{1}{13\times79}, \frac{1}{13\times41}, \frac{1}{19\times23}, \frac{1}{2\times3\times5\times7\times11\times13}$.

(176) If a mass of silver be worth £720,000 when silver is worth £4. 4*s.* per lb. av., how much would the mass be worth if silver fetched 13·75 shillings for 2·5 ounces troy?

(177) A, B and C begin playing with £1. 6*s.* each; A wins 5*s.* each game, and B loses $\frac{2}{15}$ of A's gains. After how many games will C have nothing left, and what will A then have?

(178) Reduce $\frac{222}{931}$ to a converging fraction, and give the several convergents with limits of error to each.

(179) I sold a watch for 5 guineas and thereby cleared 20 % of my money. How much % should I have gained or lost if I had sold it for $4\frac{1}{2}$ guineas?

(180) If the French 3 per cents. are at 60 when the English are at 95, the exchange between the countries being 25 fr. per £1, how much French stock in francs can be bought by selling £6000 out of the English funds?

(181) I bought silk at fr. 7·40 per metre (39·37 inches) and sold it for 6*s.* $10\frac{1}{2}d.$ per yard. Find (to 2 places) my profit or loss %, the rate of exchange being fr. 25·35 = £1.

(182) I had a cistern 5 ft., 7 in. long, 3 ft., 11 in. broad, 2 ft., $8\frac{1}{2}$ in. deep, re-lined with zinc at $8\frac{3}{4}d.$ per square foot; the plumber allowed me $\frac{7}{8}d.$ per square foot for the old zinc. How much had I to pay?

(183) Find the length of a cubical tank holding 1 ton of water, if a cubic foot of water weighs 1000 oz. av.

(184) Find in two ways $\sqrt[6]{1061520150601.}$

(185) Find in two ways to 3 places $\sqrt[6]{7\cdot\dot{3}5\dot{8}.}$

(186) If a merchantman sailing $9\frac{1}{2}$ knots an hour is chased by a gunboat steaming $10\frac{3}{4}$ knots, how far ahead must the sailing vessel be to escape into port from which she is $15\frac{1}{2}$ knots at the commencement of the chase?

(187) Divide £14. 11*s.* $8\frac{1}{2}d.$ into two parts that shall have to one another the same ratio as the sum of $2\frac{5}{8}$ and $1\frac{4}{5}$ has to their difference.

(188) Also the same ratio as the product of the two numbers has to the quotient, the greater being divided by the less.

89) **Find the side of a square field an acre in extent (to tenths yard).**

90) **The discount on a sum due 3 months hence at 5 % was . 10s. What is the sum?**

.91) **A grocer mixes 3 cwt. of tea at £16. 16s. per cwt. with 1 . at £19. 12s. At what rate per lb. must he sell the mixture s to gain 4 %?**

192) A person having invested a sum of money in the 3 per . Consols receives annually therefrom £233 after deducting income-tax of 7*d.* in the £. How much stock does he hold, how much will it be sold for, at $94\frac{1}{4}$, brokerage $\frac{1}{8}$.

193) From 122·5 grains of chlorate of potash there can be ob- ed 48 grains of oxygen gas; 16 grains of oxygen occupy a space 14·4 cubic inches. What volume of oxygen could be obtained n a ton av. of chlorate of potash?

194) Find the length of the side of a cubical tank which con- ıs 15 cwt., 7 lbs., 8 oz., of water, 1 cubic foot of which weighs 10 oz.

195) Which money sums will when decimalized yield recurring imals? and how could you get rid of the recurring figure if uired?

196) How long will it take me to travel 5 Russian versts at the e of $8\frac{1}{2}$ miles an hour?

197) Find the sum to be awarded on £87. 13*s.* 10*d.* at £7. 15*s.* *l.* %.

(198) Express $\frac{224\cdot7}{365\cdot256}$ as a continued fraction, and find the five st convergents.

(199) After the outbreak of the Prusso-French war in 1870 the ussian Government issued a 5 % war loan at 88; the French 3 r cents. stood at $65\frac{1}{2}$. State the ratio of the two rates of interest.

(200) If 9000 persons travelling each 20 miles a week pay a lroad company £900 in one week, how many persons travelling ch 30 miles weekly will give a receipt of £62,400 a year when e charge for travelling per mile is reduced one half?

(201) Find the amount of the national debt from the following sums paid as annual interest:

£3 per cent. Consolidated Annuities		£11871403	10	0
£3 per cent. Reduced	ditto	3188376	11	7
New £3 per cent.	ditto	6633792	10	10
New £3. 10*s.* per cent.	ditto	8426	2	4
New £5 per cent.	ditto	21687	9	8
New £2. 10*s.* per cent.	ditto	96176	7	0
Interest on the Government Debt to the Bank of England at 3 %		330453	0	0
Ditto to the Bank of Ireland at 3 %		78923	1	6

(202) A bar of gold weighing 8·75943 kilogs., of which $\frac{19\cdot 58}{24}$ is fine, is sent over from Paris, and sold here at £3. 17*s.* 9*d.* per standard oz., which is $\frac{22}{24}$ pure. How many francs must be remitted in payment, exchange being 25·35 fr.?

(203) Find the value of a bar of gold which weighs 11 lbs., 8 oz., 7 dwt., 12 grs., and is $\frac{21\frac{1}{2}}{24}$ pure at the rate of £3. 17*s.* 9*d.* per oz. standard.

(204) How many cubic yards of gravel will be required for a walk surrounding a rectangular lawn 200 yards long and 100 yards wide, the walk to be 3 yards wide, and the gravel 3 inches deep?

(205) Find (to two places) the side of a cubical block of cast iron weighing a ton, if iron weighs 7·2 as much as water, and a cubic foot of water weighs 1000 oz.

(206) A crown made of an alloy of copper and gold weighs 16·5 oz., while the water it displaces weighs $1\frac{1}{2}$ oz. How much copper does it contain, gold weighing 19·3 and copper 8·96 times as much as water?

(207) A grocer buys some tea at 4*s.* per lb. and some at 5*s.* 6*d.* In what proportion must he mix the two quantities so as to gain 20 % by selling the mixture at 6*s.* per lb.?

(208) Express ·583, $\cdot 58\dot{3}$, $\cdot 5\dot{8}\dot{3}$ and $\cdot\dot{5}8\dot{3}$ in the duodecimal scale to 5 places.

209) I bought £333. 6*s*. 8*d*. 3 per cent. Consols for the benefit of ɔld servant, but wished to raise his income to £25 a-year by means n investment in £50 mining shares, all paid up; the shares are $\frac{3}{4}$ premium, and the dividends are $7\frac{1}{2}$ % on the paid-up capital. w much mining stock must I buy, and what will it cost me?

210) A Lithuanian league is 9769 yards long. Find the third vergent to the fraction that an English mile is of this.

211) Express in inches the length of a French metre from the a that a metre is one ten-millionth of a quarter of the earth's umference, and that the circumference is 3·14159 times the neter 7911·7 miles.

212) State:

a. Which vulgar fractions will yield recurring and which non-recurring decimal fractions.

b. If non-recurring, how the number of places can be foretold.

c. If recurring, whether the decimal will be mixed or pure.

213) Find the value of a mass of silver weighing 15 lbs., 9 oz., dwts., 20 grs., of which $\frac{14}{17}$ is pure, at the rate of $57\frac{7}{8}d.$ per oz. ndard silver. (Standard silver contains $\frac{11 \cdot 1}{12}$ pure silver.)

214) If income-tax be 6*d*. in the £ and interest 5 %, how much I gain or lose on an income of £1200 a-year by paying the whole r's tax at the end of the third quarter instead of paying it in 4 talments at the end of each quarter?

215) Express $\frac{3}{8}$ and $419\frac{1}{7}$ in the binary, ternary, quaternary, inary, senary, septenary, octonary, nonary, decimal and duodeci- l scales.

(216) Any two numbers whose units' figures are odd, but not 5, l whose difference is a power of 10, must be prime to one another. ove this.

(217) If the wages of a woman are $\frac{4}{7}$ of the wages of a man, and would require 8 men to earn a given sum of money, how many men must be added to 5 men to earn double the money? Explain ır answer.

(218) Of the boys in a school, one-third are over 15 years of age, one-third between 10 and 15. A legacy of £100 can be exactly divided amongst them by giving 10*s.* to each boy over 15, 6*s.* 8*d.* to each between 10 and 15, and 3*s.* 4*d.* to each of the rest. How many boys are there in the school?

(219) Express the fraction $\frac{7}{25}$:

a. As a decimal fraction.

b. As a fraction having for denominator a power of 7.

c. Also a power of 4.

(220) The mint price of gold is £3. 17*s.* 10½*d.* per oz. standard. Find the smallest exact number of ounces that can be coined into an exact number of sovereigns.

(221) Find the weight of a cubical mass of iron whose edge is 2 ft., 5 in., 3 pts., if the iron is 7·157 times as heavy as water, and a cubic foot of water weighs 1000 oz. av.

(222) How high is a hill whose ascent is $1\frac{7}{8}$ miles in length, if the road rises $\frac{1}{\frac{3}{7}}$ inches in 55·25 feet?

(223) A can copy 6 pages while B copies 5, B copies 15 while C copies 12, and C can copy 4 while D copies 3; A who can write 20 pages a day receives a paper of 240 pages to copy; and after doing a quarter of it calls in B, C and D to help him. When will the work be finished?

(224) A merchant buys 1260 quarters of corn, $\frac{1}{5}$ of which he sells at a gain of 5 %, $\frac{1}{3}$ at a gain of 8 %, and the remainder at a gain of 12 %; if he had sold the whole at a gain of 10 % he would have gained £23. 2*s.* more. What was the cost price per quarter?

(225) The sum of £10552. 4*s.* 2*d.* is divided among 1000 persons in the ratio of the first thousand natural numbers. Find the share of the 150th person.

(226) The diameter of the fore wheel of a waggon is 3 ft., 6 in., that of the hind wheel 6 ft., 5 in. If two nails, one on the outside of each wheel, touch the ground together, in how many seconds (to two places) will they do so again, reckoning diameter : circumference = 1 : 3·14159, and the rate of travelling $4\frac{1}{2}$ miles an hour?

227) I bought in London \$1000 American 5-20 bonds at 87
87 × 4*s*. 6*d*. per \$100 bond). What percentage shall I get
my money if the coupons in New York fetch clear of expense
0$\frac{1}{2}$*d*. per dollar, the bonds paying 6 % ?

228) I bought in New York \$1500 5-20 bonds, brokerage $\frac{1}{8}$,
111, in (paper) currency; gold was 15$\frac{1}{2}$ premium; I paid by
upon England, and the rate of exchange was 110 (i.e. \$110
gold for 100 × 4*s*. 6*d*. payable in England); I re-sold these bonds
England at 89 (see last question), brokerage $\frac{1}{8}$. Find my profit
oss.

229) The first term of an A. P. is $\frac{1}{4}$, the common difference is $\frac{1}{2}$.
d the 50th term.

230) Suppose a debt can be discharged in a year by paying 1*s*.
1st day, 2*s*. the second, and so on. What is the amount of the
t ?

231) How many strokes do the clocks of Venice, which go on
24 o'clock, strike in a day?

(232) Find $(47{\cdot}3\dot{1}8\dot{4})^6$ to 3 places.

(233) Find the equated time for the following amounts: £50
e in 6 months, £60 in 7 months, and £80 in 10 months, interest
5 %, both by average and discount.

(234) A debt is to be paid as follows: $\frac{1}{4}$ at 2 months, $\frac{1}{8}$ at 3
nths, $\frac{1}{8}$ at 4 months, $\frac{1}{8}$ at 5 months, and the balance at 6 months.
hat is the correct equated time to pay the whole, interest at 5 %?

(235) An island is 73 miles in circumference, and 3 pedestrians
start together to travel round it in the same direction; the first
s 15, the second 17·5, and the third 10 miles a day. When will
three be again together?

(236) A owes me the following sums: £480 due in 3$\frac{1}{2}$ months,
07 due in 2 months, £577. 15*s*. due in 5 months. What sum
uld I accept as a single payment at the end of six months, reckon-
interest at 4 % ?

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237) What effect is produced on (*a*) the sum, (*b*) the difference,
two numbers, if the same quantity is added to each?

238) What effect is produced on (*a*) the sum, (*b*) the difference,
two numbers, if the same quantity is added to one and subtracted
m the other number?

(239) What effect is produced on (*a*) the product, (*b*) the quotient, of two numbers, if both numbers are multiplied by the same number?

(240) What effect is produced on the remainder, if (*a*) the divisor, (*b*) the dividend, be increased by a number not large enough to affect the quotient?

(241) What effect is produced on the remainder, if both divisor and dividend are (*a*) multiplied, (*b*) divided by the same number?

(242) State the conditions of increase in the value of a fraction, if the same number be added to both its terms.

(243) What effect is produced on the ratio, if the antecedent is multiplied and the consequent divided by the same number?

(244) What effect is produced on the square of a number, if the number is increased by a given number?

(245) Find the difference between the sum of the squares and the square of the sum of two numbers.

(246) What effect is produced on (*a*) the sum, (*b*) the difference, of two numbers, if each is multiplied by the same number?

(247) What effect is produced on (*a*) the L. C. M., (*b*) the G. C. M., (*c*) the average, of several numbers, if each is multiplied by the same number?

(248) How must a number be altered to double its reciprocal?

(249) To what limits do the *terms* of the two following series approach:

$$a.\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \ldots\ldots \qquad b.\ \tfrac{1}{7},\ \tfrac{2}{8},\ \tfrac{3}{9},\ \tfrac{4}{10},\ \ldots\ldots$$

and find the first term in each which differs from the limit by a quantity less than ·000001.

(250) To find the interest at 3 % per annum on any number of pounds for any number of days, multiply the number of pounds by twice the number of days, deduct $\frac{1}{70}$ of the product, and cut off the last two figures; the result will be the interest in pence. Shew that the error in the interest given by this rule for any time less than a year cannot exceed a shilling on every £2800 principal.

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